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Original Research Article

Imaginary Approach of Solving Analytic Functions

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Abstract

In this paper we propose a method where imaginary function "$V(x, y)$" is used for a solution of a complex variable where the real part "$U(x, y)$" is unknown. A function $f\ (z)$ is defined and differentiable at All points all points of "D". The paper utilizes the following methods in finding the unknown part "Real part $U(x, y)$" of the analytic function f(z):"The Direct Method, Milne Thompson Method and Exact Differentiable Equation Method". It was found that, out of the three methods used in finding "the real part" of the analytic function, Direct Method is more efficient as it yields results more faster, efficient and accurate.

Keywords: Analytic function, Real part, Imaginary part, complex variable, Cauchy Riemann equation, Direct Method, Milne Thomson method, Exact differentiable equation method.

INTRODUCTION

A function is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0. Also, by an analytic function we mean a function that is analytic in some domain. Hence analyticity of $f(z)$ means that $f(z)$ has a derivative at every point in some neighborhood of z_0 including itself since, by definition, is a point of all its neighborhoods [1].

The function $f(z)$is defined and continuous in some neighborhood of a point and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy–Riemann equations. Hence, if it is analytic in a domain D, those partial derivatives exist and satisfy at all points of D [1].

The Italian French Mathematician Giuseppe Luigi Lagrangia in one of his book in 1788 "Analytic Mechanics" help others to work in the field of analytic function [2]. The French Mathematicians Professor Gaspard Monge published the two first textbooks on analytic functions which are "the theory on Analytic Function" and "Lessons on the Calculus of Functions" [3]. Some mathematicians like the French mathematician Augustin- Luis Cauchy and the German mathematician Bernhard Riemann (1826-1897) and Karl Weierstrass (1815-1897) worked on analytic functions and later became the founders of complex analysis [1]. Finally Augustin-Luis Cauchy and Bernhard Riemann proposed Cauchy Riemann equation, which conclude that continuity and differentiability are necessary conditions for function to be analytic. This equation first occur in jean le Rond d'Alembert (1752). Then, Leonhard Euler connected this system to the analytic functions (1797). Cauchy (1814) used these equations to construct his theory of function. Bernhard Riemann introduced the concept on the integral as it used in basic calculus courses and made important contributions to differential equation and also called Riemannian-geometry which is the mathematical foundation of Einstein theory of relativity [1] and his dissertation appeared on 1851 which is called Riemann Dissertation [4].

In 2012 Dr Nirav Vyas reported the application of analytic function firstly in the electrostatic field that satisfied laplace's equation $\nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial Y^2} = 0$, where$\emptyset$ is the real part of some analytic function $f(z) = \emptyset(x, y) + i\ v\ (x, y)$. He also reported in heat flow problem where the heat equation is reduces to $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. $T(x, y)$ is called the heat potential, which is the real part af complex heat potential $f(z) =$

$T(x,y) + i\,\psi\,(x,y)$. He later presented the existence of analytic fuinction in Fluid Flow $f(z) = \varphi(x,y) + i\,\psi\,(x,y)$. Where $f(z)$ is called complex potential of the flow φ is called the velocity potential and ψ is called stream function. Dr Niav also reported the application of analytic function in Magnetic Field, where the magnetic potential φ is the real part of some complex variable analytic function $f(z) = \varphi(x,y) + i\,\psi\,(x,y)$ [5].

Kundan Kumar (2015) proposed KK's method to find an Analytic Function if either "U" or "V" is given without using Cauchy Riemann equation for complex variable [2]. M Yaswanth (2018) modified the work of Kundan Kumar and proposed a method to find analytic function with the help of real or imaginary function [3]. This paper presents a modified method of [2, 3] in finding the solution of analytic function using imaginary approach where the real part is unknown.

1. METHODOLOGY AND RESULT

A. DIRECT METHOD

To find the real part of an analytic function where the imaginary part "$V(x,y)$" is given, we apply the following steps:

A function f(z) is said to be analytic function if:

$(i)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ (ii)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial u}$

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Step 2 : from (i) $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$ we have $U = \int \frac{\partial v}{\partial y}\, dx$

Step 3 : from $(ii)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial u}$ we have $U = \int -\frac{\partial v}{\partial x}\, dy$

Step 4 : add Step 2 and Step 3 by adding both similar and non similar terms

Step 5 : Produce $f(z) = U(x,y) + iV(x,y)$

Example 1: Find the analytic function whose imaginary part is given by

$V(x,y) = 3x^2y - y^3 + 6xy$

Solution: from the analytic function $(i)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ (ii)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial u}$

And by applying Step 1 to step 5 we have

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

$V(x,y) = 3x^2y - y^3 + 6xy$

$\frac{\partial v}{\partial x} = 6xy + 6y$ (with respect to x)

$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$ (with respect to y)

Step 2 : from (i) U= $\int \frac{\partial v}{\partial y}$ dx

$$U = \int (3x^2 - 3y^2 + 6x)\, dx$$

$U = x^3 - 3y^2x + 3x^2 + c$

Step 3 : we have U= $\int -\frac{\partial v}{\partial x}$ dy

U= $\int -(6xy + 6y)$ dy

$U = -3xy^2 - 3y^2 + c$

Step 4 : add Step 2 and Step 3 by adding both similar and non similar terms

$U(x,y) = x^3 - 3y^2x + 3x^2 + c$

$U(x,y) = -3xy^2 - 3y^2 + c$

Similar terms: $-3xy^2, c$

Non similar terms: $x^3, -3y^2, 3x^2$

$U(x,y) = x^3 + 3x^2 - 3xy^2 - 3y^2 + c$

Step 5 : f(z)= $(x^3 + 3x^2 - 3xy^2 - 3y^2 + c) + i(3x^2y - y^3 + 6xy)$

Example 2: Find the analytic function whose imaginary part is given by $V(x,y): tan^{-1}(\frac{y}{x})$

Solution:

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Let $m = \frac{y}{x} = yx^{-1}, \frac{\partial M}{\partial x} = -yx^{-2}$

$\frac{\partial v}{\partial x} = tan^{-1}M \times \frac{\partial M}{\partial x}$

$= \frac{1}{1+M^2} \times -yx^{-2}$

$= \frac{1}{1+(\frac{y}{x})^2} \times -yx^{-2}$

$= -yx^{-2} \times \frac{x^2}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$

Let $m = \frac{y}{x} = yx^{-1}, \frac{\partial M}{\partial y} = x^{-1}$

$\frac{\partial v}{\partial y} = tan^{-1}m \times \frac{\partial M}{\partial y}$

$= \frac{1}{1+M^2} \times x^{-1}$

$= \frac{1}{1+(\frac{y}{x})^2} \times -yx^{-2}$

$= x^{-1} \times \frac{x^2}{x^2+y^2}$

$= \frac{x^1}{x^2+y^2}$

$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$

Step 2 : from (i) U= $\int \frac{\partial v}{\partial y}\, dx$

U= $\int \frac{x}{x^2+y^2}$ dx

Let m = $x^2 + y^2$, $\frac{\partial M}{\partial x} = 2x$, $\partial x = \frac{\partial M}{2x}$

U= $\int \frac{x}{m} \times \frac{\partial m}{2x}$

$= \int \frac{1}{m} \times \frac{\partial m}{2}$

$= \frac{1}{2}\int \frac{1}{m}\partial m$

$= \frac{1}{2} log\ m + c$

U = $\frac{1}{2}\ log\ (x^2 + y^2) + c$

Step3

U= $\int -\frac{\partial v}{\partial x}\, dy$

U= $\int -(\frac{-y}{x^2+y^2})\, \partial y$

Let m = $x^2 + y^2$, $\frac{\partial M}{\partial y} = 2y$, $\partial y = \frac{\partial M}{2y}$

U= $\int \frac{y}{m} \times \frac{\partial M}{2y}$

$= \frac{1}{2}\int \frac{1}{m}\partial m$

$= \frac{1}{2} log\ m + c$

U= $\frac{1}{2} log\ (x^2 + y^2)$+c

Step 4 : add Step 2 and Step 3 by adding both similar and non similar terms

$U(x,y)$= $\frac{1}{2}log\ x^2+y^2$+c
$U(x,y)$ = $\frac{1}{2}log\ x^2+y^2$+c
Similar terms: - $\frac{1}{2}log\ x^2+y^2$,c
Non similar terms: 0
U(x,y)= $\frac{1}{2}log\ x^2+y^2$+c
Step 5 : *f(z)=*$\frac{1}{2}$ *log(x²+y²)+c+i* $tan^{-1}(\frac{y}{x})$

Example 3: Find the analytic function whose imaginary part is given by $V(x,y)\ =\ 3x^2y-y^3$
Solution
V(x,y) = $3x^2y-y^3$
$\frac{\partial v}{\partial x}$= $6xy$ (with respect to x)
$\frac{\partial v}{\partial y}$= $3x^2-\ 3y^2$(with respect to y)
U= $\int\frac{\partial v}{\partial y}dx$
U= $\int 3x^2-3y^2\ dx$
U = $x^3-3y^3\ x$ + c
U= $\int-\frac{\partial v}{\partial x}$dy
U=$\int -6xy\ dy$
U = $-3xy^2$+c
Step 4 : add Step 2 and Step 3 by adding both similar and non similar terms
$U(x,y)=\ x^3-3y^2x\ +\ c$
$U(x,y)$ = $-3xy^2$+c
Similar terms: $-3xy^2$,c
Non similar terms: x^3
$U(x,y)=\ x^3-3y^2x\ +\ c$
Step 5 : f(z)= x³-3y²x+c+i(3x²y-y³)

B. MILNE THOMSON METHOD

A complex Variable function given by$f\ (z)$: $U(x,y)+iV(x,y)$ is said to be analytic if it satisfies the following conditions:
$(i)\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ (ii)\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial u}$

If the imaginary part "V" is given, the real part "U" can be found by respecting the following steps
Since$f(z)$: $U(x,y)+iV(x,y)$
$\frac{\partial f(z)}{\partial x}=\frac{\partial u}{\partial x}\ +\ i\frac{\partial v}{\partial x}$
Since $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$
f'(z)=$\frac{\partial u}{\partial x}\ +\ i\frac{\partial v}{\partial x}$
Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$
Step 2: Substitute $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in f'(z)
Step 3: put $x\ =\ z$ and $y\ =\ 0$ in $f\ '(z)$
Step 4: find$\int f'dz$ and put $Z\ =\ x+iy$
Step 5: arrange your equation

Example 1: Find the analytic function whose imaginary part is given by $V(x,y)\ =\ 3x^2y-y^3+6xy$
Solution: we know that
$$f'(z)=\frac{\partial u}{\partial x}\ +\ i\frac{\partial v}{\partial x}$$
Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$
$V(x,y)\ =\ 3x^2y-y^3+6xy$
$\frac{\partial v}{\partial x}$= $6xy+6y$(with respect to x)
$\frac{\partial v}{\partial y}$= $3x^2-3y^2+6x$ (with respect to y)
Step 2: Substitute $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in $f'\ (z)$
f'(z) = $3x^2-3y^2+6x$+$i(6xy+6y)$
Step 3: put $x\ =\ z$ and $y\ =\ 0$ in $f'\ (z)$
$F'\ (z)\ =\ 3z^2-0^2+6z+i(6z(0)+6(0))$
$f'(z)\ =\ 3z^2+6z$
Step 4 : find $\int\ f'(z)\ dz$
$$\int\ f'(z)\ =\int(3z^2+6z)\ dz$$
$f(z)\ =\ z^3+3z^2+c$
Put $z\ =\ x+iy$
$f(z)=\ (x+iy)^3+3(x+iy)^2+c$
$=(x+iy)^2(x+iy)+3(x^2+2ixy-y^2)+c$
$=(x^2+2ixy-y^2)(x+iy)+3(x^2+2ixy-y^2)+c$
$=\ x^2-3x^2y-3y^2+3x^2+3ix^2y+6ixy-iy^3+c$
$f(z)=\ U(x,y)+iV(x,y)$
U (x, y) = x²-3x²y-3y²+3x²+C

Example 1: Find the analytic function whose imaginary part is given by
$V(x,y)\ =\ e^x(x\cos y-y\sin y)$
Solution:
$V(x,y)\ =\ e^x(x\cos y-y\sin y)$
We know that
f'(z)=$\frac{\partial u}{\partial x}\ +\ i\frac{\partial v}{\partial x}$
Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$
$$\frac{\partial v}{\partial x}=e^x\cos y+xe^x\cos y-ye^x\sin y$$
$=e^x(\cos y+x\cos y-y\sin y)$
$$\frac{\partial v}{\partial y}=-xe^x\sin y-ye^x\cos y-e^x\sin y$$
$=e^x(-x\sin y-y\cos y-\sin y)$
Step 2: Substitute $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in $f'(z)$
$f'(z)$=$e^x(-x\sin y-y\cos y-\sin y)\ +i[e^x(x\cos y-y\sin y)]$
Step 3: put x=z and y= 0 in $f'(z)$
$f'(z)=e^z(-z\sin 0-0\cos 0-\sin 0)+i[e^z\ (z\cos 0-0\sin 0)]$
$f'(z)$=$e^z(0)+i[e^z(1+z)]$
$f'(z)=i(e^z+e^zz)$
Step 4 : find $\int f'(z)\,dz$
$\int f'(z)=\int i(e^z+e^zz)\ dz$

By using integral by part
$f(z)$=$i(e^z+e^zz-e^z)+c$
$f(z)$=$i(e^zz)+c$
Put $=x+iy$,
$i(x+iy)e^{x+iy})+c$
$=i(x+iy)e^x.e^{iy}+c$
$=(ix-y)e^x(cosy+isiny)+c$
$=ixe^xcosy-xe^xsiny-ye^xcosy-iye^xsiny+c$
$=[e^x(-xsiny-ycosy)+c]+ie^x(xcosy-ysiny)$

$U(x,y) = e^x(-xsiny - ycosy)$
$V(x,y) = e^x(xcosy - ysiny)$

C. EXACT DIFFERENTIAL EQUATION METHOD

Step 1: Find $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $(ii)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial u}$

Step2: $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$ we have

U= $\int \frac{\partial v}{\partial y} dx + g(y)$

Step 3 :$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx + g(y)\right]$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx\right] + g'(y)$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx\right] + g'(y) = -\frac{\partial v}{\partial x}$

Step 4 :

$g'(y) = -\frac{\partial v}{\partial x} - \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx\right]$

$g'(y) = -\left\{\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx\right]\right\}$

Step 5 : find $\int g'(y)$

Step 6 : substitute step 5 in step 2

Example 1: Find the analytic function whose imaginary part is given by $V(x,y) = 3x^2y - y^3 + 6xy$

Solution: from the analytic function $(i)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $(ii)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial u}$

And by applying Step 1 to step 5 we have

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

$V(x,y) = 3x^2y - y^3 + 6xy$

$\frac{\partial v}{\partial x} = 6xy + 6y$ (with respect to x)

$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$ (with respect to y)

Step 2: from (i) $U = \int \frac{\partial v}{\partial y} dx + g(y)$

$U = \int (3x^2 - 3y^2 + 6x)\, dx + g(y)$

$U = x3 - 3y^2x + 3x^2 + g(y)$

Step 3: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial y} = -6xy + g'(y) = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial y} = -6xy + g'(y) = -6xy - 6y$

Step 4 :$g'(y) = -6xy - 6y + 6xy$

$g'(y) = -6y$

Step 5:

$\int g'(y) = \int -6y$

$g(y) = -3y^2$

Step 6 : substitute step 5 in step 2

$U(x,y) = x^2 - 3x^2y + 3x^2 - 3y^2$

f(z) =($x^2 - 3x^2y + 3x^2 - 3y^2$)+i($3x^2y - y^3 + 6xy$)

Example 1: Find the analytic function whose imaginary part is given by $V(x,y) = cosx\ sinhy$

Solution

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

$V(x,y) = \cos x \sinh y$

$\frac{\partial v}{\partial x} = \sin x \sinh y$ (with respect to x)

$\frac{\partial v}{\partial y} = \cos x \cosh y$ (with respect to y)

Step 2 : from (i) $U = \int \frac{\partial v}{\partial y} dx + g(y)$

$U = \int \cos x \cosh y\, dx + g(y)$

$U = \sin x \cosh y + g(y)$

Step 3 :$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial y} = \sin x \sinh y + g'(y) = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial y} = \sin x \sinh y + g'(y) = -(-\sin x \sinh y)$

$g'(y) = \sin x \sinh y - \sin x \sinh y$

$g'(y) = 0$

Step 4:

$g'(y) = 0$

Step 5: $\int g'(y) = \int 0$

$g(y) = C$

Step 6 : substitute step 5 in step 2

$U(x,y) = \sin x \cosh y + C$

f(z) =($\sin x \cosh y$ + C)+i ($\cos x \sinh y$)

CONCLUSION

This research was able to find the real part of the analytic function with the help of imaginary real part with all the three methods mentioned. But the direct Method is much easier and accurate than the remaining methods.

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