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Original Research Article

Flexural Rigidity Influence on Dynamic Response of Orthoropic Rectangular Plate Resting on Constant Elastic Foundation

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Abstract

This research article considers the flexural rigidity influence along both x and y axes on the dynamic response of orthotropic rectangular plate resting on constant elastic foundation with elastic end conditions. The orthotropic rectangular plate model is a coupled fourth order partial differential equation having variables and singular coefficients. The solutions to this model are arrived at by reconstructing the fourth order partial differential. This partial differential equation model is converted to a set of coupled second order ordinary differential equations by using a special technique adopted by Shadnam *et al*., [11]. This set of second order ordinary differential equations is then reduced using modified asymptotic method of Struble. The closed form solution is evaluated, resonance conditions are obtained and the results are showed in plotted curves to depict the influence of flexural rigidities along both x and y axes on the dynamic response of orthotropic rectangular plate resting on constant elastic foundation with elastic end conditions for both cases of moving distributed mass and moving distributed force.

Keywords: Bi-parametric Elastic Foundation, Flexural rigidity, Moving Distributed Masses, Transvered Displacement, Resonance.

1. INTRODUCTION

Flexural rigidity plays a crucial role in understanding the dynamic response of orthotropic rectangular structures that are supported on an elastic foundation. The term "flexural rigidity" refers to a material's resistance to bending or deformation under the action of applied load. In the case of orthotropic rectangular structures, the flexural rigidity is influenced by the material properties and the dimensions of the structure. When an orthotropic rectangular structure is supported on a constant elastic foundation, its dynamic response is significantly affected by flexural rigidity of the structure. The dynamic response refers to the behavior of the structure when subjected to dynamic loads or vibrations. The flexural rigidity of the structure determines the stiffness of the structure in resisting bending deformations. A higher flexural rigidity implies greater resistance to bending, resulting in a stiffer structure. Conversely, a lower flexural rigidity allows for more bending and deformations in the structure. The dynamic response of the structure is disturbed by the flexural rigidity in several ways. Firstly, a higher flexural rigidity results to a higher natural frequency of the structure. The term “natural frequency” is the frequency at which the structure vibrates without any external forces. A stiffer structure has a higher natural frequency, meaning it will vibrate at a higher frequency. Secondly, the flexural rigidity influences the mode shapes of the structure. Mode shapes refer to the patterns of vibration exhibited by the structure at different frequencies. A higher flexural rigidity can result in more rigid mode shapes, where the structure exhibits minimal deformation during vibration. On the other hand, a lower flexural rigidity can lead to flexible mode shapes, where the structure undergoes significant deformations during vibration. Lastly, the flexural rigidity affects the damping characteristics of the structure. Damping refers to the dissipation of energy during vibration. A higher flexural rigidity typically results in higher damping, as the structure is better able to resist deformations and dissipate energy. Conversely, a lower flexural rigidity may result in lower damping, leading to more energy being retained in the structure during vibration.

Many researchers in the fields of applied mathematics and mechanics have worked tiredlessly on plate model most especially on orthotropic rectangular plate model. Some of these researchers include: Hermon [1] extended the work of Warbuton [2] to analyse the free vibration of rectangular orthotropic plates having either clamped or simply supported conditions using the Rayleigh method. Hosseini and Fadaee [3] proposed an exact solution for free flexural vibration of rectangular thick plates using third order shear deformation plate theory. Werfalli and Karoud [4] conducted a free vibration analysis of rectangular plates using Galerkin based finite element method. Mama [5] studied and even proposed a solution offree harmonic equation of simply supported plates using Galerkin-Vlasov method. Hatiegan and *et al.*, [6] analysed thin clamped plates of different geometric forms using finite element method. Benamar and *et al.*, [7] examined the effects of large vibration amplitudes on the mode shapes and natural frequencies of thin isotropic plates. Alfano and Pagnotta [8] performed a suitable approximate relationships, relating the resonance frequencies to the elastic constants of isotropic thin plates. Awodola and Adeoye [9] carried out study the vibration of orthotropic rectangular plates on variable elastic Pasternak foundation with clamped end conditions. Ugural [10] and Okafor and Oguaghamba [11] adoptied Ritz and Galarkin methods to solve isotropic and orthotropic plate problems. Awodola and Omolofe studied the response of concentrated moving masses of elastically supported rectangular plates on Winkler elastic foundation by the method of separation of variable. Adeoye and Awodola [12] investigated the dynamic behaviour of moving distributed masses of orthotropic rectangular plate with clamped-clamped boundary conditions on constant elastic foundation. Hu, *et al.*, [13] purported vibration solutions to orthotropic rectangular plates by making use of symplectic geometry method. Adeoye and Adeloye [14] assessed the dynamic characteristics of orthotropic rectangular plate under the influence of moving distributed masses on variable elastic foundation using the variable separable.

In all the aforementioned researches, no works explicitly discussed the influence of flexural rigidities along both x and y axes on the dynamic response of orthoropic rectangular plate. In this research, the influence of flexural rigidities along both x and y axes on the dynamic response of orthoropic rectangular plate resting on constant elastic foundation with elastic end conditions will be investigated.

2. GOVERNING EQUATION

The transverse displacement $W(x,y,t)$ of orthotropic rectangular plates that rests on a bi-parametric elastic foundation and traversed by distributed mass$M_r$traversing with constant velocity k_r along a straight line parallel to the x-axis issuing from point y = φ on the y-axis with flexural rigidities D_x and D_y is governed by the fourth order partial differential equation given as:

$$
\begin{aligned}
&D_x \frac{\partial^4}{\partial x^4} W(x,y,t) + 2\beta \frac{\partial^4}{\partial x^2 \partial y^2} W(x,y,t) + D_y \frac{\partial^4}{\partial y^4} W(x,y,t) + \mu \frac{\partial^2}{\partial t^2} W(x,y,t) \\
&-\rho h R_0 [\frac{\partial^4}{\partial x^2 \partial t^2} W(x,y,t) + \frac{\partial^4}{\partial y^2 \partial t^2} W(x,y,t)] + K_0 W(x,y,t) - G_0 [\frac{\partial^2}{\partial x^2} \\
&W(x,y,t) + \frac{\partial^2}{\partial y^2} W(x,y,t)] - \sum_{r=1}^{N} [M_r g H(x - k_r t) H(y - \varphi) - M_r (\frac{\partial^2}{\partial t^2} W(x,y,t) \\
&+2c_r \frac{\partial^2}{\partial x \partial t} W(x,y,t) + c_r^2 \frac{\partial^2}{\partial x^2} W(x,y,t)) H(x - k_r t) H(y - \varphi) W(x,y,t)] = 0
\end{aligned} \tag{1}
$$

Where D_x and D_y are the flexural rigidities of the plate along x and y axes respectively.

$$D_x = \frac{E_x h^3}{12(1-\nu_x \nu_y)}, \quad D_y = \frac{E_y h^3}{12(1-\nu_x \nu_y)}, \quad B = D_x D_y + \frac{G_o h^3}{6}$$

E_x and E_y are the Young's moduli along x and y axes respectively, G_o is the rigidity modulus, ν_x and ν_y are Poisson's ratios for the material such that $E_x \nu_y = E_y \nu_x$, ρ is the mass density per unit volume of the plate, h is the plate thickness, t is the time, x and y are the spatial coordinates in x and y directions respectively, R_o is the rotatory inertia correction factor, K_o is the foundation constant and g is the acceleration due to gravity, H(.) is the Heaviside function.

Re-expressing equation (2), one obtains

$$
\begin{aligned}
&\mu \frac{\partial^2}{\partial t^2} W(x,y,t) + \mu \kappa_n^2 W(x,y,t) = \rho h R_0 [\frac{\partial^4}{\partial x^2 \partial t^2} W(x,y,t) + \frac{\partial^4}{\partial y^2 \partial t^2} W(x,y,t)] \\
&-2\beta \frac{\partial^4}{\partial x^2 \partial y^2} W(x,y,t) - D_x \frac{\partial^4}{\partial x^4} W(x,y,t) - D_y \frac{\partial^4}{\partial y^4} W(x,y,t) - K_0 W(x,y,t) \\
&+\mu \kappa_n^2 W(x,y,t) + G_0 [\frac{\partial^2}{\partial x^2} W(x,y,t) + \frac{\partial^2}{\partial y^2} W(x,y,t)] + \sum_{r=1}^{N} [M_r g H(x - k_r t) \\
&H(y - \varphi) - M_r (\frac{\partial^2}{\partial t^2} W(x,y,t) \ + 2k_r \frac{\partial^2}{\partial x \partial t} W(x,y,t) + k_r^2 \frac{\partial^2}{\partial x^2} W(x,y,t)) H(x - k_r t)
\end{aligned} \tag{2}
$$

$H(y-\varphi)W(x,y,t)]$

Which can be represented further as:

$$\frac{\partial^2}{\partial t^2}W(x,y,t)+\kappa_n^2 W(x,y,t)=R_0[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)]-\frac{2B}{\mu}\frac{\partial^4}{\partial x^2\partial y^2} W(x,y,t)-\frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}W(x,y,t)-\frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}W(x,y,t)+[\kappa_n^2-\frac{K_0}{\mu}]W(x,y,t)+\frac{G_0}{\mu}[\frac{\partial^2}{\partial x^2} W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)]+\sum_{r=1}^{N}[\frac{M_r g}{\mu}H(x-k_r t)\mathrm{H}(y-\varphi)-\frac{M_r}{\mu}(\frac{\partial^2}{\partial t^2}W(x,y,t)+2k_r \frac{\partial^2}{\partial x\partial t}W(x,y,t)+k_r^2\frac{\partial^2}{\partial x^2}W(x,y,t))H(x-k_r t)H(y-\varphi)W(x,y,t)] \quad (3)$$

Where κ_n^2 is the natural frequencies, $n=1,2,3,\ldots$

The initial conditions, without any loss of generality, is taken as:

$$W(x,y,t)=0=\frac{\partial}{\partial t}W(x,y,t) \quad (4)$$

3. Analytical Approximate Solution

To obtain an expression for the solution of equation (4), one applies technique of Shadnam *et al.*, [11] which requires that the deflection of the plates be in series form as

$$W(x,y,t)=\sum_{n=1}^{N}\delta_n(x,y)\xi_n(t) \quad (5)$$

Where $\delta_n(x,y)=\delta_{ni}(x)\delta_{nj}(y)$ and

$$\delta_{ni}(x)=\sin\frac{\gamma_{ni}}{L_x}x+A_{ni}\cos\frac{\gamma_{ni}}{L_x}x+B_{ni}\sinh\frac{\gamma_{ni}}{L_x}x+C_{ni}\cosh\frac{\gamma_{ni}}{L_x}x$$

$$\delta_{ni}(y)=\sin\frac{\gamma_{nj}}{L_y}y+A_{nj}\cos\frac{\gamma_{nj}}{L_y}y+B_{nj}\sinh\frac{\gamma_{nj}}{L_y}y+C_{nj}\cosh\frac{\gamma_{nj}}{L_y}y \quad (6)$$

The right hand side of equation (4) when written in series form takes the form:

$$\sum_{n=1}^{\infty}R_0[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)]-\frac{2\beta}{\mu}\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t)-\frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}W(x,y,t)-\frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}W(x,y,t+[\kappa_n^2-\frac{K_0}{\mu}]W(x,y,t)+\frac{G_0}{\mu}[\frac{\partial^2}{\partial x^2}W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)]+\sum_{r=1}^{N}[\frac{M_r g}{\mu}H(x-k_r t)\mathrm{H}(y-\varphi)-\frac{M_r}{\mu}(\frac{\partial^2}{\partial t^2}W(x,y,t)+2k_r\frac{\partial^2}{\partial x\partial t}W(x,y,t)+k_r^2\frac{\partial^2}{\partial x^2} W(x,y,t))H(x-k_r t)H(y-\varphi)W(x,y,t)]=\sum_{n=1}^{N}\delta_n(x,y)\varrho_n(t) \quad (7)$$

Multiplying both sides of equation (8) by $\delta_m(x,y)$, integrating on area A of the plate and considering the orthogonality of $\delta_m(x,y)$, one gets:

$$\varrho_n(t)=\frac{1}{\eta^*}\sum_{n=1}^{\infty}\int_A[R_0(\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t))-\frac{2\beta}{\mu}\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t)-\frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}W(x,y,t)-\frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}W(x,y,t+(\kappa_n^2-\frac{K_0}{\mu})W(x,y,t)+\frac{G_0}{\mu}(\frac{\partial^2}{\partial x^2}W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t))+\sum_{r=1}^{N}[\frac{M_r g}{\mu}H(x-k_r t)H(y-\varphi)-\frac{M_r}{\mu}(\frac{\partial^2}{\partial t^2}W(x,y,t)+2k_r\frac{\partial^2}{\partial x\partial t} W(x,y,t)+k_r^2\frac{\partial^2}{\partial x^2}W(x,y,t))H(x-k_r t)H(y-\varphi)W(x,y,t)]\delta_m(x,y)dA \quad (8)$$

and zero when $n\neq m$

Where

$$\eta^*=\int_A\delta_n^2(x,y)dA \quad (9)$$

Making use of equation (6), equation (8), taking into account equation (4), can be written as:

$$\delta_n(x,y)[\ddot{\xi}_n(t)+\kappa_n^2\xi_n(t)]=\frac{\delta_n(x,y)}{\eta^*}\sum_{q=1}^{\infty}\int_A[R_0(\frac{\partial^2\delta_q(x,y)}{\partial x^2}\delta_m(x,y)\ddot{\xi}_q(t)+\frac{\partial^2\delta_q(x,y)}{\partial y^2}$$
$$\delta_m(x,y)\ddot{\xi}_q(t))-\frac{2\beta}{\mu}\frac{\partial^2\delta_q(x,y)}{\partial x^2\partial y^2}\delta_m(x,y)\xi_q(t)-\frac{D_x}{\mu}\frac{\partial^4\delta_q(x,y)}{\partial x^4}\delta_m(x,y)\xi_q(t)-\frac{D_y}{\mu}$$
$$\frac{\partial^4\delta_q(x,y)}{\partial y^4}\delta_m(x,y)\xi_q(t)+(\kappa_n^2-\frac{K_0}{\mu})\delta_q(x,y)\delta_m(x,y)\xi_q(t)+\frac{G_0}{\mu}(\frac{\partial^2\delta_q(x,y)}{\partial x^2}\delta_m(x,y)$$
$$\xi_q(t)+\frac{\partial^2\delta_q(x,y)}{\partial y^2}\delta_m(x,y)\xi_q(t))+\sum_{r=1}^{N}(\frac{M_rg}{\mu}\delta_m(x,y)H(x-k_rt)H(y-s)-\frac{M_r}{\mu}(\Psi_q(x,y)$$

$$\delta_m(x,y)\ddot{\xi}_q(t)+2k_r\frac{\partial\delta_q(x,y)}{\partial x}\delta_m(x,y)\dot{\xi}_q(t)+k_r^2\frac{\partial^2\delta_q(x,y)}{\partial x^2}\delta_m(x,y)\xi_q(t))H(x-k_rt)$$
$$H(y-\varphi))]dA \tag{10}$$

When equation (10) is simplified further, one obtains:

$$\ddot{\xi}_n(t)+\kappa_n^2\xi_n(t)=\frac{1}{\eta^*}\sum_{q=1}^{\infty}\int_A[R_0(\frac{\partial^2\delta_q(x,y)}{\partial x^2}\delta_m(x,y)\ddot{\xi}_q(t)+\frac{\partial^2\delta_q(x,y)}{\partial y^2}\delta_m(x,y)\ddot{\xi}_q(t)$$
$$)-\frac{2\beta}{\mu}\frac{\partial^2\delta_q(x,y)}{\partial x^2\partial y^2}\delta_m(x,y)\xi_q(t)-\frac{D_x}{\mu}\frac{\partial^4\delta_q(x,y)}{\partial x^4}\delta_m(x,y)\xi_q(t)-\frac{D_y}{\mu}\frac{\partial^4\delta_q(x,y)}{\partial y^4}\delta_m(x,y)$$
$$\xi_q(t)+(\kappa_n^2-\frac{K_0}{\mu})\delta_q(x,y)\delta_m(x,y)Q_q(t)+\frac{G_0}{\mu}(\frac{\partial^2\delta_q(x,y)}{\partial x^2}\delta_m(x,y)\xi_q(t)+\frac{\partial^2\delta_q(x,y)}{\partial y^2}$$
$$\delta_m(x,y)\xi_q(t))+\sum_{r=1}^{N}(\frac{M_rg}{\mu}\delta_m(x,y)H(x-k_rt)H(y-s)-\frac{M_r}{\mu}(\delta_q(x,y)\delta_m(x,y)$$

$$\ddot{\xi}_q(t)+2k_r\frac{\partial\delta_q(x,y)}{\partial x}\delta_m(x,y)\dot{\xi}_q(t)+k_r^2\frac{\partial^2\delta_q(x,y)}{\partial x^2}\delta_m(x,y)\xi_q(t))H(x-k_rt)H(y-\varphi))]dA \tag{11}$$

The system of equations in equation (11) is a set of coupled ordinary differential equations Making use of Fourier series representation, the Heaviside functions take the form

$$H(x-k_rt)=\frac{1}{4}+\frac{1}{\pi}\sum_{a=1}^{N}\frac{\sin(2a+1)\pi(x-k_rt)}{2a+1},0<x<1 \qquad (12)$$

r

$$H(y-\varphi)=\frac{1}{4}+\frac{1}{\pi}\sum_{k=1}^{N}\frac{\sin(2k+1)\pi(y-\varphi)}{2k+1},0<y<1 \qquad (13)$$

Substituting equations (12) and (13) into equation (11) and simplifying, one obtains:

$$\ddot{\xi}_n(t)+\kappa_n^2\xi_n(t)-\frac{1}{\eta^*}\sum_{q=1}^{\infty}[R_0\varepsilon_0\ddot{\xi}_q(t)-\frac{2\beta}{\mu}\varepsilon_1\xi_q(t)-\frac{D_x}{\mu}\varepsilon_2\xi_q(t)-\frac{D_y}{\mu}\varepsilon_3\xi_q(t)+$$
$$(\kappa_n^2-\frac{K_0}{\mu})\varepsilon_4\xi_q(t)+\frac{G_0}{\mu}\varepsilon_5\xi_q(t)-\sum_{r=1}^{N}\frac{M_r}{\mu}((\varepsilon_6+\frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_1^*\frac{\cos(2a+1)\pi k_rt}{2a+1}-$$
$$\sum_{a=1}^{\infty}\tau_2^*\frac{\sin(2a+1)\pi k_rt}{2a+1})(\sum_{k=1}^{\infty}\tau_3^*\frac{\cos(2k+1)\pi\varphi}{2k+1}-\sum_{k=1}^{\infty}\tau_4^*\frac{\sin(2k+1)\pi\varphi}{2k+1})+\frac{1}{4\pi}$$
$$(\sum_{a=1}^{\infty}\tau_5^*\frac{\cos(2a+1)\pi k_rt}{2a+1}-\sum_{a=1}^{\infty}\tau_6^*\frac{\sin(2a+1)\pi k_rt}{2a+1})+\frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_7^*\frac{\cos(2k+1)\pi\varphi}{2k+1}-$$
$$\sum_{k=1}^{\infty}\tau_8^*\frac{\sin(2k+1)\pi\varphi}{2k+1}))\ddot{\xi}_q(t)+2k_rt(\varepsilon_7+\frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_9^*\frac{\cos(2a+1)\pi k_rt}{2a+1}-\sum_{a=1}^{\infty}\tau_{10}^*$$
$$\frac{\sin(2a+1)\pi k_rt}{2a+1})(\sum_{k=1}^{\infty}\tau_{11}^*\frac{\cos(2k+1)\pi\varphi}{2k+1}-\sum_{k=1}^{\infty}\tau_{12}^*\frac{\sin(2k+1)\pi\varphi}{2k+1})+\frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_{13}^* \tag{14}$$
$$\frac{\cos(2a+1)\pi k_rt}{2a+1}-\sum_{a=1}^{\infty}\tau_{14}^*\frac{\sin(2a+1)\pi k_rt}{2a+1})+\frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{15}^*\frac{\cos(2k+1)\pi\varphi}{2k+1}-\sum_{k=1}^{\infty}\tau_{16}^*$$
$$\frac{\sin(2k+1)\pi\varphi}{2k+1}))\dot{\xi}_q(t)+k_r^2(\varepsilon_8+\frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_{17}^*\frac{\cos(2a+1)\pi k_rt}{2a+1}-\sum_{a=1}^{\infty}\tau_{18}^*\frac{\sin(2j+1)\pi k_rt}{2j+1}$$
$$)(\sum_{k=1}^{\infty}\tau_{19}^*\frac{\cos(2k+1)\pi\varphi}{2k+1}-\sum_{k=1}^{\infty}\tau_{20}^*\frac{\sin(2k+1)\pi\varphi}{2k+1})+\frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_{21}^*\frac{\cos(2a+1)\pi k_rt}{2a+1}$$
$$-\sum_{a=1}^{\infty}\tau_{22}^*\frac{\sin(2a+1)\pi k_rt}{2a+1})+\frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{23}^*\frac{\cos(2k+1)\pi\varphi}{2k+1}-\sum_{k=1}^{\infty}\tau_{24}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}))$$
$$\xi_q(t))]=\sum_{q=1}^{\infty}\sum_{r=1}^{N}\frac{M_rg}{\mu\eta^*}\delta_m(k_rt)\delta_m(\varphi)$$

Which is the transformed equation that governs the problem of an orthotropic rectangular plate resting on constant bi-parametric elastic foundation.

Where

$$\tau_0 = \int_A [\frac{\partial^2}{\partial x^2}\delta_q(x,y)\delta_m(x,y) + \frac{\partial^2}{\partial y^2}\delta_q(x,y)\delta_m(x,y)]dA \quad (15)$$

$$\varepsilon_1 = \int_A \frac{\partial^2}{\partial x^2}[\frac{\partial^2}{\partial x^2}\delta_q(x,y)]\delta_m(x,y)dA \quad (16)$$

$$\varepsilon_2 = \int_A \frac{\partial^4}{\partial x^4}[\delta_q(x,y)]\delta_m(x,y)dA \quad (17)$$

$$\varepsilon_3 = \int_A \frac{\partial^4}{\partial y^4}[\delta_q(x,y)]\delta_m(x,y)dA \quad (18)$$

$$\varepsilon_4 = \int_A \delta_q(x,y)\delta_m(x,y)dA \quad (19)$$

$$\varepsilon_5 = \int_A [\frac{\partial^2}{\partial x^2}\delta_q(x,y) + \frac{\partial^2}{\partial y^2}\delta_q(x,y)]\delta_m(x,y)dA \quad (20)$$

$$\varepsilon_6 = \frac{1}{16}\int_A \delta_q(x,y)\delta_m(x,y)dA \quad (21)$$

$$\tau_1^* = \int_A \delta_q(x,y)\delta_m(x,y)\sin(2\mathrm{a}+1)\pi x dA \quad (22)$$

$$\tau_2^* = \int_A \delta_q(x,y)\delta_m(x,y)\cos(2\mathrm{a}+1)\pi x dA \quad (23)$$

$$\tau_3^* = \int_A \delta_q(x,y)\delta_m(x,y)\sin(2k+1)\pi y dA \quad (24)$$

$$\tau_4^* = \int_A \delta_q(x,y)\delta_m(x,y)\cos(2k+1)\pi y dA \quad (25)$$

$$\tau_5^* = \tau_1^*, \quad \tau_6^* = \tau_2^*, \quad \tau_7^* = \tau_3^*, \quad \tau_8^* =_4^* \quad (26)$$

$$\varepsilon_7 = \frac{1}{16}\int_A \frac{\partial}{\partial x}\delta_q(x,y)\delta_m(x,y)dA \quad (27)$$

$$\tau_9^* = \int_A \frac{\partial}{\partial x}(\delta_q(x,y))\delta_m(x,y)\sin(2\mathrm{a}+1)\pi x dA \quad (28)$$

$$\tau_{10}^* = \int_A \frac{\partial}{\partial x}(\delta_q(x,y))\delta_m(x,y)\cos(2\mathrm{a}+1)\pi x dA \quad (29)$$

$$\tau_{11}^* = \int_A \frac{\partial}{\partial x}(\delta_q(x,y))\delta_m(x,y)\sin(2k+1)\pi y dA \quad (30)$$

$$\tau_{12}^* = \int_A \frac{\partial}{\partial x}\delta_q(x,y)\delta_m(x,y)\cos(2k+1)\pi y dA \quad (31)$$

$$\tau_{13}^* = \tau_9^*, \quad \tau_{14}^* = \tau_{10}^*, \quad \tau_{15}^* = \tau_{11}^*, \quad \tau_{16}^* = \tau_{12}^* \quad (32)$$

$$\varepsilon_8 = \frac{1}{16}\int_A \frac{\partial^2}{\partial x^2}(\delta_q(x,y))\delta_m(x,y)dA \quad (33)$$

$$\tau_{17}^* = \int_A \frac{\partial^2}{\partial x^2}(\delta_q(x,y))\delta_m(x,y)\sin(2\mathrm{a}+1)\pi x dA \quad (34)$$

$$\tau_{18}^* = \int_A \frac{\partial^2}{\partial x^2}(\delta_q(x,y))\delta_m(x,y)\cos(2\mathrm{a}+1)\pi x dA \quad (35)$$

$$\tau_{19}^* = \int_A \delta_q(x,y)\delta_m(x,y)\sin(2k+1)\pi y dA \quad (36)$$

$$\tau_{20}^* = \int_A \frac{\partial^2}{\partial x^2}(\delta_q(x,y))\delta_m(x,y)\cos(2k+1)\pi y dA \quad (37)$$

$\tau_{21}^* = \tau_{17}^*,\quad \tau_{22}^* = \tau_{18}^*,\quad \tau_{23}^* = \tau_{19}^*,\quad \tau_{24}^* = \tau_{20}^*$ (38)

$\delta_m(x,y)$ is assumed to be the products of functions $\delta_{jm}(x)\delta_{wm}(y)$ which are the beam functions in the directions of x and y axes respectively. That is

$\delta_m(x,y) = \delta_{jm}(x)\delta_{wm}(y)$ (39)

Where,

$$\delta_{jm}(x) = \sin\lambda_{jm}x + A_{jm}\cos\lambda_{jm}x + B_{jm}\sinh\lambda_{jm}x + C_{jm}\cosh\lambda_{jm}x$$

$$\delta_{wm}(y) = \sin\lambda_{wm} + A_{wm}\cos\lambda_{wm}y + B_{wm}\sinh\lambda_{wm}y + C_{wm}\cosh\lambda_{wm}y \quad (40)$$

Where A_{jm}, B_{jm}, C_{jm}, A_{jm}, B_{jm} and C_{jm} are constants determined by the boundary conditions while δ_{jm} and δ_{wm} are called the mode frequencies

Where

$\lambda_{jm} = \frac{\zeta_{jm}}{L_x},\quad \lambda_{wm} = \frac{\zeta_{wm}}{L_y}$ (41)

Considering a unit mass, equation (19) can be re-expressed as

$$\ddot{\xi}_n(t) + \kappa_n^2\xi_n(t) - \frac{1}{\eta^*}\sum_{q=1}^{\infty}\Big[R_0\varepsilon_0\ddot{\xi}_q(t) - \frac{2\beta}{\mu}\varepsilon_1\xi_q(t) - \frac{D_x}{\mu}\varepsilon_2\xi_q(t) - \frac{D_y}{\mu}\varepsilon_3\xi_q(t)$$
$$+(\kappa_n^2 - \frac{K_0}{\mu}\varepsilon_4)\xi_q(t) + \frac{G_0}{\mu}\varepsilon_5\xi_q(t) - \alpha\sigma((\varepsilon_6 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_1^*\frac{\cos(2a+1)\pi kt}{2a+1} -$$
$$\sum_{a=1}^{\infty}\tau_2^*\frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty}\tau_3^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_4^*\frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_5^*$$
$$\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_6^*\frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_7^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_8^*$$

$$\frac{\sin(2k+1)\pi\varphi}{2k+1}))\ddot{\xi}_q(t) + 2k_r(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_9^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_{10}^*\frac{\sin(2a+1)\pi k_r t}{2a+1}$$
$$)(\sum_{k=1}^{\infty}\tau_{11}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{12}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_{13}^*\frac{\cos(2a+1)\pi k_r t}{2a+1} -$$
$$\sum_{a=1}^{\infty}\tau_{14}^*\frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{15}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{16}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}))\dot{\xi}_q(t)$$

$$+{k_r}^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_{17}^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_{18}^*\frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty}\tau_{19}^*$$
$$\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{20}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_{21}^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_{22}^*$$
$$\frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{23}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{24}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}))\xi_q(t))\Big] \quad (42)$$
$$= \sum_{q=1}^{\infty}\frac{Mg}{\mu\eta^*}\delta_m(k_r t)\delta_m(\varphi)$$

Equation (43) is the fundamental equation of the rectangular plate problem.

Where,

$\alpha = \frac{M}{\mu\sigma},\quad \sigma = L_x L_y$ (43)

$\delta_m(ct) = \sin\phi_m(t) + A_m\cos\phi_m(t) + B_m\sinh\phi_m(t) + C_m\cosh\phi_m(t)$ (44)

$\delta_m(s) = \sin\iota_m + A_m\cos\iota_m + B_m\sinh\iota_m + C_m\cosh\iota_m$ (45)

$\phi_m = \frac{\gamma_m k_r}{L_x},\quad \iota_m = \frac{\gamma_{m\varphi} s}{L_y}$ (46)

3.1 Orthotropic Rectangular Plate Traversed by a Moving Force

In moving force problem in mechanics, the motion of the structure or body is being influenced by an external force that is continuously changing or moving. This force which is represented by the moving load is assumed being only transferred to the structure. In this case, the inertia effect is negligible. Setting $\alpha = 0$ in the fundamental equation (42), one obtains:

$$\ddot{\xi}_n(t) + (1 - \frac{\varepsilon_4}{\mu\eta^*})\kappa_n^2\xi_n(t) - \frac{1}{\mu\eta^*}[\mu R_0\varepsilon_0\ddot{\xi}_n(t) - 2\beta\varepsilon_1\xi_n(t) - D_x\varepsilon_2\xi_n(t) - D_y\varepsilon_3\xi_n(t) - K_0\varepsilon_4\xi_n(t) + G_0\varepsilon_5\xi_n(t) + \sum_{q=1,q\neq n}^{\infty}(\mu R_0\varepsilon_0\ddot{\xi}_q(t) - 2\beta\varepsilon_1\xi_q(t) - D_x\varepsilon_2\xi_q(t) - D_y\varepsilon_3\xi_q(t) + (\mu\kappa_q^2 - K_0\varepsilon_4)\xi_q(t) + G_0\varepsilon_5\xi_q(t))] = \frac{Mg}{\mu\eta^*}\delta_m(k_r t)\delta_m(\varphi) \quad (47)$$

Which is further be simplified as:

$$\ddot{\xi}_n(t) + \Omega_n^2\xi_n(t) - \Psi[\mu R_0\varepsilon_0\ddot{\xi}_n(t) - 2\beta\varepsilon_1\xi_n(t) - D_x\varepsilon_2\xi_n(t) - D_y\varepsilon_3\xi_n(t) - K_0\varepsilon_4\xi_n(t) + G_0\varepsilon_5\xi_n(t) + \sum_{q=1,q\neq n}^{\infty}(\mu R_0\varepsilon_0\ddot{\xi}_q(t) - 2\beta\varepsilon_1\xi_q(t) - D_x\varepsilon_2\xi_q(t) - D_y\varepsilon_3\xi_q(t) + (\mu\Omega_q^2 - K_0\varepsilon_4)\xi_q(t) + G_0\varepsilon_5\xi_q(t))] = \Psi Mg\delta_m(k_r t)\delta_m(\varphi) \quad (48)$$

Where $\Omega_n^2 = (1 - \frac{\varepsilon_4}{\mu\eta^*})\kappa_n^2,\ \Psi = \frac{1}{\mu\eta^*}$

Expanding and re-arranging equation (48), one gets:

$$[1 - \Psi\mu R_0\varepsilon_0]\ddot{\xi}_n(t) + (\Omega_n^2 - \Psi J_6)\xi_n(t) - \Psi\sum_{q=1,q\neq n}^{\infty}(\mu R_0\varepsilon_0\ddot{\xi}_q(t) - 2\beta\varepsilon_1\xi_q(t) - D_x\varepsilon_2\xi_q(t) - D_y\varepsilon_3\xi_q(t) + (\mu\kappa_q^2 - K_0\varepsilon_4)\xi_q(t) + G_0\varepsilon_5\xi_q(t)) = \Psi Mg\delta_m(k_r t)\delta_m(\varphi) \quad (49)$$

Simplifying further, one obtains:

$$\ddot{\xi}_n(t) + \frac{(\Omega_n^2 - \Psi J_6)}{[1 - \Psi\mu R_0\varepsilon_0]}\xi_n(t) + \frac{\Psi}{[1 - \Psi\mu R_0\varepsilon_0]}\sum_{q=1,q\neq n}^{\infty}(\mu R_0\varepsilon_0\ddot{\xi}_q(t) - 2\beta\varepsilon_1\xi_q(t) - D_x\varepsilon_2\xi_q(t) - D_y\varepsilon_3 Q_q(t) + (\mu\kappa_q^2 - K_0\varepsilon_4)\xi_q(t) + G_0\varepsilon_5\xi_q(t)) = \frac{\Psi Mg}{[1 - \Psi\mu R_0 T_0]}\delta_m(k_r t)\delta_m(\varphi) \quad (50)$$

Where,

$$J_6 = -2\beta\varepsilon_1 - D_x\varepsilon_2 - D_y\varepsilon_3 - K_0\varepsilon_4 + G_0\varepsilon_5 \quad (51)$$

For any arbitrary ratio Ψ, defined as:

$\Psi^* = \frac{\Psi}{1+\Psi}$, one obtains

$$\Psi = \frac{\Psi^*}{1 - \Psi^*} = \Psi^* + o(\Psi^{*2}) + \ldots$$

For only $o(\Psi^*)$, one obtains

$$\Psi = \Psi^*$$

On application of binomial expansion,

$$\frac{1}{1 - \Psi^*\mu R_0\varepsilon_0} = 1 + \Psi^*\mu R_0 T_0 + o(\Psi^{*2}) + \ldots \quad (52)$$

On putting equation (52) into equation (50), one obtains:

$$\ddot{\xi}_n(t) + (\Omega_n^2 - \Psi^* J_6)(1 + \Psi^*\mu R_0\varepsilon_0 + o(\Psi^{*2}) + \ldots)\xi_n(t) + \Psi^*(1 + \Psi^*\mu R_0\varepsilon_0 + o(\Psi^{*2}) + \ldots)\sum_{q=1,q\neq n}^{\infty}(\mu R_0\varepsilon_0\ddot{\xi}_q(t) - 2\beta\varepsilon_1\xi_q(t) - D_x\varepsilon_2\xi_q(t) - D_y\varepsilon_3\xi_q(t) + (\mu\kappa_q^2 - K_0\varepsilon_4)\xi_q(t) + G_0\varepsilon_5\xi_q(t)) = \Psi^* aMg(1 + \Psi^*\mu R_0 T_0 + o(\Psi^{*2}) + \ldots)\delta_m(k_r t)\delta_m(\varphi) \quad (53)$$

Retaining only $o(\Psi^*)$, equation (54) becomes:

$$\ddot{\xi}_n(t) + [\Omega_n^2(1 + \Psi^*\mu R_0 T_0) - \Psi^* J_6]\xi_n(t) + \Psi^*\sum_{q=1,q\neq n}^{\infty}(\mu R_0 T_0\ddot{\xi}_q(t) - 2\beta\varepsilon_1\xi_q(t) - D_x\varepsilon_2\xi_q(t) - D_y\varepsilon_3\xi_q(t) + (\mu\kappa_q^2 - K_0\varepsilon_4)\xi_q(t) + G_0\varepsilon_5\xi_q(t)) = \Psi^* Mg\delta_m(k_r t)\delta_m(\varphi) \quad (54)$$

Which is simplified further as:

$$\ddot{\xi}_n(t) + \Omega_n^2\xi_n(t) + \Psi^*\sum_{q=1,q\neq n}^{\infty}(\mu R_0\varepsilon_0\ddot{\xi}_q(t) - 2\beta\varepsilon_1\xi_q(t) - D_x\varepsilon_2\xi_q(t) - D_y\varepsilon_3\xi_q(t) (\mu\kappa_q^2 - K_0\varepsilon_4)\xi_q(t) + G_0\varepsilon_5\xi_q(t)) = \Psi^* Mg\delta_m(k_r t)\delta_m(\varphi) \quad (55)$$

Where,
$J_7 = [\Omega_n^2(1 + \Psi^*\mu R_0\varepsilon_0) - \Psi^* J_6]$ (56)

Using Struble's technique, one obtains
$\Omega_{nn} = \Omega_n - (\frac{\Omega_n^2 - J_7}{2\Omega_n})$ (57)

Represents the modified frequency for moving force problem.
Using equation (58), the homogeneous part of equation (55) can be written as:
$\ddot{\xi}_n(t) + \Omega_{nn}^2 \xi_n(t) = 0$ (58)

Hence, the entire equation (56) gives:
$\ddot{\xi}_n(t) + \Omega_{nn}^2 \xi_n(t) = \Psi^* Mg\delta_m(k_r t)\delta_m(k_r)$ (59)

On solving equation (59) one obtains:

$$\xi_n(t) = \frac{Mg\Psi^*\delta_m(\varphi)}{\Omega_{nn}(\phi_m^4 - \Omega_{nn}^4)}[(\phi_m^2 + \Omega_{nn}^2)(\phi_m \sin\Omega_{nn}t - \Omega_{nn}\sin\alpha_m t) - A_m\Omega_{nn}(\phi_m^2 + \Omega_{nn}^2) (\cos\phi_m t - \cos\Omega_{nn}t) - B_m(\phi_m^2 - \Omega_{nn}^2)(\phi_m \sin\Omega_{nn}t - \Omega_{nn}\sinh\phi_m t) + C_m\Omega_{nn}(\phi_m^2 - \Omega_{nn}^2)(\cosh\phi_m t - \cos\Omega_{nn}t)] \quad (60)$$

Making use of equation (5) one obtains:

$$W(x,y,t) = \sum_{jm=1}^{\infty}\sum_{wm=1}^{\infty} \frac{Mg\Psi^*\delta_m(\varphi)}{\Omega_{nn}(\phi_m^4 - \Omega_{nn}^4)}[(\phi_m^2 + \Omega_{nn}^2)(\phi_m \sin\Omega_{nn}t - \Omega_{nn}\sin\alpha_m t) - A_m\Omega_{nn} (\phi_m^2 + \Omega_{nn}^2)(\cos\phi_m t - \cos\Omega_{nn}t) - B_m(\phi_m^2 - \Omega_{nn}^2)(\phi_m \sin\Omega_{nn}t - \Omega_{nn}\sinh\phi_m t) + C_m\Omega_{nn} (\phi_m^2 - \Omega_{nn}^2)(\cosh\phi_m t - \cos\Omega_{nn}t)](\sin\frac{\zeta_{jm}}{L_x}x + A_{jm}\cos\frac{\zeta_{jm}}{L_x}x + B_{jm}\sinh\frac{\zeta_{jm}}{L_x}x + C_{jm}\cosh\frac{\zeta_{jm}}{L_x}x)(\sin\frac{\zeta_{wm}}{L_y}y + A_{wm}\cos\frac{\zeta_{wm}}{L_y}y + B_{wm}\sinh\frac{\zeta_{wm}}{L_y}y + C_{wm}\cosh\frac{\zeta_{wm}}{L_y}y) \quad (61)$$

Represents the transverse displacement response to a moving force problem of orthotropic rectangular plate.

3.2 Orthtropic Rectangular Plate Traversed by a Moving Mass

In moving mass problem, the system or body is subjected to an external force or forces as it moves. The behaviour of the system is influenced by the interaction occurred between the applied forces and the system's mass, which results in various changes, effects and phenomena. That is to say, the weight and as well as inertia forces are transferred to the moving load. That is the inertia effect is not negligible. That is, $\alpha \neq 0$ and so it is expedient to solve the entire equation (42).

To solve this equation, one make use of an analytical approximate method. This method is known as an approximate analytical method of Struble. The homogeneous part of equation (42) shall be replaced by a free system operator defined by the modified frequency ξ_{nn}. Thus, the entire equation becomes:

$$\ddot{\xi}_n(t) + \Omega_{nn}^2\xi_n(t) + \alpha\Theta^*\sum_{q=1}^{\infty}[(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_1^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}E_2^*\frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty}\tau_3^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_4^*\frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_5^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_6^*\frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_7^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_8^*\frac{\sin(2k+1)\pi\varphi}{2k+1}))\ddot{\xi}_q(t) + 2k_r(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_9^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_{10}^*\frac{\sin(2a+1)\pi k_r t}{2a+1}$$

$$)(\sum_{k=1}^{\infty}\tau_{11}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{j=1}^{\infty}\tau_{12}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_{13}^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_{14}^*\frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{15}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{16}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}))\dot{\xi}_q(t) + {k_r}^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty}\tau_{17}^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_{18}^*\frac{\sin(2a+1)\pi k_r t}{2a+1}) (\sum_{k=1}^{\infty}\tau_{19}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{20}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_{21}^*\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty}\tau_{22}^*\frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{23}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{24}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}))\xi_q(t)] = \sum_{q=1}^{\infty}\frac{Mg}{\mu\eta^*}\delta_m(k_r t)\delta_m(\varphi) \quad (62)$$

Where $\Theta^* = \frac{\sigma}{\eta^*}$

On expanding and simplifying equation (62), one obtains:

$$\ddot{\xi}_n(t) + \Omega_{nn}^2\xi_n(t) + \alpha\Theta^*[(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_1^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_2^*$$
$$\frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_3^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_4^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_5^*$$
$$\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_6^* \frac{\sin(2a+1)\pi k_r}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_7^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_8^*$$
$$\frac{\sin(2k+1)\pi\varphi}{2k+1}))\ddot{\xi}_n(t) + 2k_r(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_9^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{10}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}$$

$$)(\sum_{k=1}^{\infty} \tau_{11}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{j=1}^{\infty} \tau_{12}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{13}^* \frac{\cos(2a+1)\pi k_r t}{2a+1}$$
$$- \sum_{a=1}^{\infty} \tau_{14}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_{15}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{16}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}$$
$$))\dot{\xi}_n(t) + k_r^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_{17}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{18}^* \frac{\sin(2a+1)\pi k_r t}{2a+1})$$
$$(\sum_{k=1}^{\infty} \tau_{19}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{20}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{21}^* \frac{\cos(2a+1)\pi k_r t}{2a+1}$$
$$- \sum_{a=1}^{\infty} \tau_{22}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_{23}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{24}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}$$

$$))\xi_n(t)] + \alpha\Theta^* \sum_{q=1, q\neq n}^{\infty} [(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_1^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_2^*$$
$$\frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_3^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_4^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_5^*$$
$$\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_6^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_7^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_8^*$$
$$\frac{\sin(2k+1)\pi\varphi}{2k+1}))\ddot{\xi}_q(t) + 2k_r(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_9^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{10}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}$$
$$)(\sum_{k=1}^{\infty} \tau_{11}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{12}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{13}^* \frac{\cos(2a+1)\pi k_r t}{2a+1}$$
$$- \sum_{a=1}^{\infty} \tau_{14}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_{15}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{16}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}$$
$$))\dot{\xi}_q(t) + k_r^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_{17}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{18}^* \frac{\sin(2a+1)\pi k_r t}{2a+1})$$

$$(\sum_{k=1}^{\infty} \tau_{19}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{20}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{21}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{22}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_{23}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{24}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}))\xi_q(t)] = \Psi^* M g \delta_m(k_r t)\delta_m(\varphi) \quad (63)$$

On further rearrangements and simplifications, one obtains:

$$(1 + \alpha\Theta^*(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} \tau_1^* \frac{\cos(2j+1)\pi k_r t}{2j+1} - \sum_{a=1}^{\infty} \tau_2^* \frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_3^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_4^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_5^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_6^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_7^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_8^* \frac{\sin(2k+1)\pi\varphi}{2k+1})))\ddot{\xi}_n(t) + 2k_r\alpha\Theta^*(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_9^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{10}^* \frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_{11}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{12}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{13}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{14}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_{15}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{16}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}$$

$$))\dot{\xi}_n(t) + (\Omega_{nn}^2 + \alpha\Theta^* k_r^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_{17}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{18}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) (\sum_{k=1}^{\infty} \tau_{19}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{20}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{21}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{22}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_{23}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{24}^* \frac{\sin(2k+1)\pi\varphi}{2k+1})))\xi_n(t) + \alpha\Theta^* \sum_{q=1, q\neq n}^{\infty} [(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_1^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{j=1}^{\infty} \tau_2^* \frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_3^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_4^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_5^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_6^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_7^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_8^*$$

$$\frac{\sin(2k+1)\pi\varphi}{2k+1}))\ddot{\xi}_q(t) + 2k_r(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_9^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{10}^* \frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_{11}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{12}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{13}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{14}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_{15}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{16}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}))\dot{\xi}_q(t) + k_r^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_{17}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{18}^* \frac{\sin(2a+1)\pi k_r t}{2a+1})$$

$$(\sum_{k=1}^{\infty} \tau_{19}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{20}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{21}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{22}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} \tau_{23}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{24}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}))\xi_q(t)] = \Psi^* Mg\delta_m(k_r t)\delta_m(\varphi) \quad (64)$$

On further expression of equation (65), one obtains:

$$\ddot{\xi}_n(t) + 2k_r\alpha\Theta^*(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_9^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{10}^* \frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_{11}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{12}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{13}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{14}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + (\sum_{k=1}^{\infty} \tau_{15}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{16}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}))\dot{\xi}_n(t) +$$

$$(\Omega_{nn}^2(1 - \alpha\Theta^*(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_1^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_2^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) (\sum_{k=1}^{\infty} \tau_3^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_4^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_5^* \frac{\cos(2j+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_6^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + (\sum_{k=1}^{\infty} \tau_7^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_8^* \frac{\sin(2k+1)\pi\varphi}{2k+1})))) +$$

$$k_r^2\alpha\Theta^*(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_{17}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{18}^* \frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_{19}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{20}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{21}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} E_{22}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + (\sum_{k=1}^{\infty} \tau_{23}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{24}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}))\xi_n(t)$$

$$+\alpha\Theta^* \sum_{q=1,q\neq n}^{\infty} [(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_1^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_2^* \frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_3^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_4^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_5^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_6^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + (\sum_{k=1}^{\infty} \tau_7^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_8^* \frac{\sin(2k+1)\pi\varphi}{2k+1}))\ddot{\xi}_q(t) +$$

$$2k_r(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_9^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{10}^* \frac{\sin(2v+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_{11}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{12}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{13}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{14}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + (\sum_{k=1}^{\infty} \tau_{15}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{16}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}))\dot{\xi}_q(t) + k_r^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{a=1}^{\infty} \tau_{17}^*$$

$$\frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} E_{18}^* \frac{\sin(2a+1)\pi k_r t}{2a+1})(\sum_{k=1}^{\infty} \tau_{19}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{20}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}) + \frac{1}{4\pi}(\sum_{a=1}^{\infty} \tau_{21}^* \frac{\cos(2a+1)\pi k_r t}{2a+1} - \sum_{a=1}^{\infty} \tau_{22}^* \frac{\sin(2a+1)\pi k_r t}{2a+1}) + \frac{1}{4\pi} (\sum_{k=1}^{\infty} \tau_{23}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{24}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}))\xi_q(t)] = \Psi^* Mg\delta_m(k_r t)\delta_m(\varphi) \quad (65)$$

Applying the modified asymptotic method of Struble, equation (65) can be re-expressed as:

$$\ddot{\xi}_n(t) + \omega_n^2 \xi_n(t) = 0 \quad (66)$$

for the homogeneous case

Hence, the entire equation becomes:

$$\ddot{\xi}_n(t) + \omega_n^2 \xi_n(t) = \Psi^* M g \delta_m(k_r t)\delta_m(\varphi) \quad (67)$$

Where,

$$\omega_n = \Omega_{nn} - \frac{1}{2\Omega_{nn}}\left(\Omega_{nn}^2 \alpha \Theta^*\left(\varepsilon_6 + \frac{1}{4\pi}\left(\sum_{k=1}^{\infty} \tau_7^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_8^* \frac{\sin(2k+1)\pi\varphi}{2k+1}\right)\right) - k_r^2 \alpha \Theta^*\left(\varepsilon_8 + \frac{1}{4\pi}\left(\sum_{k=1}^{\infty} \tau_{23}^* \frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty} \tau_{24}^* \frac{\sin(2k+1)\pi\varphi}{2k+1}\right)\right)\right) \quad (68)$$

Which gives the modified frequency representing the frequency of the free system.
Rewriting equation (67), one obtains

$$\ddot{\xi}_n(t) + \omega_n^2 \xi_n(t) = \Psi^* M g \delta_m(s)[\sin\phi_m(t) + A_m \cos\phi_m t + B_m \sinh\phi_m t + C_m \cosh\phi_m t] \quad (69)$$

Making use of the procedures applied to solve equation (59) earlier, one obtains

$$\xi_n(t) = \frac{\Psi^* M g \delta_m(\varphi)}{\_n(\phi_m^4 - \omega_n^4)}[(\phi_m^2 + \omega_n^2)(\phi_m \sin\omega_n t - \omega_n \sin\phi_m t) - A_m \omega_n(\phi_m^2 + \omega_n^2)(\cos\phi_m t - \cos\omega_n t) - B_m(\phi_m^2 - \omega_n^2)(\phi_m \sin\omega_n t - \omega_n \sinh\phi_m t) + C_m \omega_n(\phi_m^2 - \omega_n^2)(\cosh\phi_m t - \cos\omega_n t)] \quad (70)$$

On making use of equation (5)

$$W(x,y,t) = \sum_{jm=1}^{\infty} \sum_{wm=1}^{\infty} \frac{\Psi^* M g \delta_m(\varphi)}{\_n(\phi_m^4 - \omega_n^4)}[(\phi_m^2 + \omega_n^2)(\phi_m \sin\omega_n t - \omega_n \sin\phi_m t) - A_m \omega_n(\phi_m^2 + \omega_n^2)(\cos\phi_m t - \cos\omega_n t) - B_m(\phi_m^2 - \omega_n^2)(\phi_m \sin\omega_n t - \omega_n \sinh\phi_m t) + C_m \omega_n(\phi_m^2 - \omega_n^2)(\cosh\phi_m t - \cos\omega_n t)]\left(\sin\frac{\zeta_{jm}}{L_x}x + A_{jm}\cos\frac{\zeta_{jm}}{L_x}x + B_{jm}\sinh\frac{\zeta_{jm}}{L_x}x + C_{jm}\cosh\frac{\zeta_{jm}}{L_x}x\right)\left(\sin\frac{\zeta_{wm}}{L_y}y + A_{wm}\cos\frac{\zeta_{wm}}{L_y}y + B_{wm}\sinh\frac{\zeta_{wm}}{L_y}y + C_{wm}\cosh\frac{\zeta_{wm}}{L_y}y\right) \quad (71)$$

Represents the transverse displacement response to a moving mass of an orthotropic rectangular plate.

4. ILLUSTRATIVE EXAMPLES

4.1 Orthotropic Rectangular Plate with Clamped at All Edges

For an orthotropic rectangular plate with clamped at all edges, the boundary conditions are given byplate is simply supported at $x = 0$, $x = L_x$ and elastically supported at $y = 0$, $y = L_y$ the boundary conditions are expressed below:

$$W(0,y,t) = 0 = W(L_x,y,t), \quad W(x,y,t) = 0 = W(x,L_y,t) \quad (72)$$

$$\frac{\partial}{\partial x}W(0,y,t) = 0 = \frac{\partial}{\partial x}W(L_x,y,t), \quad \frac{\partial}{\partial y}W(x,0,t) = 0 = \frac{\partial}{\partial x}W(0,L_y,t) \quad (73)$$

Thus, for the normal modes

$$\zeta_{jm}(0) = 0 = \zeta_{jm}(L_x), \quad \zeta_{wm}(0) = 0 = \zeta_{wm}(L_y) \quad (74)$$

$$\frac{\partial \zeta_{jm}(0)}{\partial x} = 0 = \frac{\partial \zeta_{jm}(L_x)}{\partial x}, \quad \frac{\partial \zeta_{wm}(0)}{\partial y} = 0 = \frac{\partial \zeta_{wm}(L_y)}{\partial y} \quad (75)$$

For uniformity, our initial conditions take the form of

$$W(x,y,0) = 0 = \frac{\partial W(x,y,0)}{\partial t} \quad (76)$$

Adopting the boundary conditions in equations (72) to (75) and the initial conditions given by equation (76), it can be shown that:

$$A_{jm} = \frac{\sinh\zeta_{jm} - \sin\zeta_{jm}}{\cos\zeta_{jm} - \cosh\zeta_{jm}} = \frac{\cos\zeta_{jm} - \cos\zeta_{jm}}{\sin\zeta_{jm} + \sinh\zeta_{jm}} \quad (77)$$

$$A_{wm} = \frac{\sinh\zeta_{wm} - \sin\zeta_{wm}}{\cos\zeta_{wm} - \cosh\zeta_{wm}} = \frac{\cos\zeta_{wm} - \cos\zeta_{wm}}{\sin\zeta_{wm} + \sinh\zeta_{wm}} \quad (78)$$

In the same vein, we have:

$$A_m = \frac{\sinh\zeta_m - \sin\zeta_m}{\cos\zeta_m - \cosh\zeta_m} = \frac{\cos\zeta_m - \cos\zeta_m}{\sin\zeta_m + \sinh\zeta_m} \quad (79)$$

and

$$B_{jm} = -1, \quad B_{wm} = -1, \quad \Rightarrow B_m = -1 \quad C_{jm} = -A_{jm}, \quad C_{wm} = -A_{wm}, \quad \Rightarrow C_m = -A_m \quad (80)$$

From equation (79), one obtains

$$\cos\zeta_m \cosh\zeta_m = 1 \quad (81)$$

represents the frequency equation for the dynamical problem, such that

$$\zeta_1 = 4.73004, \quad \zeta_2 = 7.85320, \quad \zeta_3 = 10.9951, \ldots \quad (82)$$

In reference to equations (79), (80) and (82) in equations (61) and (71), one gets the displacement response to a moving force and a moving mass of orthotropic rectangular plate on bi-parametric elastic foundation respectively.

4.2 Orthotropic Rectangular Plate with Clamped Elastic Boundary Conditions

For the clamped end, both deflection and slope varnish. Thus, when the orthotropic plate is both clamped at $x = 0$ and $y = 0$ and elastically supported at $x = L_x$ and $y = L_y$, the conditions are of the form:

$$W(0, y, t) = W'(0, y, t) \quad (83)$$

at the end $x = 0$ and

$$W''(L_x, 0, t) - \varphi_1 W'(L_x, 0, t) = 0 = W'''(L_x, 0, t) + \varphi_2 W(L_x, 0, t) \quad (84)$$

at the end $x = L_x$

In the same, we have

$$W(0, y, t) = W'(0, y, t) \quad (85)$$

at the end $y = 0$ and

$$W''(0, L_y, t) - \varphi_1 W'(0, L_y, t) = 0 = W'''(0, L_y, t) + \varphi_2 W(0, L_y, t) \quad (86)$$

at the end $y = L_y$

Thus, for normal modes, we have:

$$\zeta_{jm}(0) = 0 = \zeta'_{jm}(L_x), \quad \zeta_{wm}(0) = 0 = \zeta'_{wm}(L_y) \quad (87)$$

at the end $x = 0$ and $y = 0$ and

$$\begin{aligned} \zeta''_{jm}(L_x) - \varphi_1 \zeta'_{jm}(L_x) = 0 = \zeta'''_{jm}(L_x) + \varphi_2 \zeta_{jm}(L_x) \\ \zeta''_{wm}(L_y) - \varphi_1 \zeta'_{wm}(L_y) = 0 = \zeta'''_{wm}(L_y) + \varphi_2 \zeta_{wm}(L_y) \end{aligned} \quad (88)$$

Also

$$\begin{aligned} \zeta''_{m}(L_x) - \varphi_1 \zeta'_{m}(L_x) = 0 = \zeta'''_{m}(L_x) + \varphi_2 \zeta_{m}(L_x) \\ \zeta''_{m}(L_y) - \varphi_1 \zeta'_{m}(L_y) = 0 = \zeta'''_{m}(L_y) + \varphi_2 \zeta_{m}(L_y) \end{aligned} \quad (89)$$

Using equations (87) and (88), it can be shown that at $x = 0$,

$$A_{jm} = -C_{jm}, \quad B_{jm} = -1, \quad A_{wm} = -C_{wm}, \quad B_{wm} = -1 \quad (90)$$

Also,

$$A_m = -C_m, \quad B_m = -1 \quad (91)$$

and

$$B_{jm} = \frac{\frac{\zeta_{jm}}{L_x}[\sin\zeta_{jm} + \sinh\zeta_{jm}] + \varphi_1[\cos\zeta_{jm} - \cosh\zeta_{jm}]}{\frac{\zeta_{jm}}{L_x}[\cos\zeta_{jm} + \cosh\zeta_{jm}] - \varphi_1[\cos\zeta_{jm} + \cos\zeta_{jm}]} = \frac{\frac{\zeta_{jm}^3}{L_x^3}[\cos\zeta_{jm} - \sin\zeta_{jm}] + \varphi_2[\sinh\zeta_{jm} - \sin\zeta_{jm}]}{-\frac{\zeta_{jm}^3}{L_x^3}[\sin\zeta_{jm} - \sinh\zeta_{jm}] + \varphi_1[\cos\zeta_{jm} - \cosh\zeta_{jm}]} = -C_{jm} \quad (92)$$

Similarly, we have:

$$B_{wm} = \frac{\frac{\zeta_{wm}}{L_y}[\sin\zeta_{wm}+\sinh\zeta_{wm}]+\varphi_1[\cos\zeta_{wm}-\cosh\zeta_{wm}]}{\frac{\zeta_{wm}}{L_y}[\cos\zeta_{wm}+\cosh\zeta_{wm}]-\varphi_1[\cos\zeta_{wm}+\cos\zeta_{wm}]} = \frac{\frac{\zeta_{wm}^3}{L_y^3}[\cos\zeta_{wm}-\sin\zeta_{wm}]+\varphi_2[\sinh\zeta_{wm}-\sin\zeta_{wm}]}{-\frac{\zeta_{wm}^3}{L_y^3}[\sin\zeta_{wm}-\sinh\zeta_{wm}]+\varphi_1[\cos\zeta_{wm}-\cosh\zeta_{wm}]} = -C_{wm} \quad (93)$$

In the same vein, we have:

$$B_m = \frac{\frac{\zeta_m}{L_x}[\sin\zeta_m+\sinh\zeta_m]+\varphi_1[\cos\zeta_m-\cosh\zeta_m]}{\frac{\zeta_m}{L_x}[\cos\zeta_m+\cosh\zeta_m]-\varphi_1[\cos\zeta_{jm}+\cos\zeta_m]} = \frac{\frac{\zeta_m^3}{L_x^3}[\cos\zeta_m-\sin\zeta_m]+\varphi_2[\sinh\zeta_m-\sin\zeta_m]}{-\frac{\zeta_m^3}{L_x^3}[\sin\zeta_m-\sinh\zeta_m]+\varphi_1[\cos\zeta_m-\cosh\zeta_m]} = -C_m \quad (94)$$

and

$$B_m = \frac{\frac{\zeta_m}{L_y}[\sin\zeta_m+\sinh\zeta_m]+\varphi_1[\cos\zeta_m-\cosh\zeta_m]}{\frac{\zeta_m}{L_y}[\cos\zeta_m+\cosh\zeta_m]-\varphi_1[\cos\zeta_{jm}+\cos\zeta_m]} = \frac{\frac{\zeta_m^3}{L_y^3}[\cos\zeta_m-\sin\zeta_m]+\varphi_2[\sinh\zeta_m-\sin\zeta_m]}{-\frac{\zeta_m^3}{L_y^3}[\sin\zeta_m-\sinh\zeta_m]+\varphi_1[\cos\zeta_m-\cosh\zeta_m]} = -C_m \quad (95)$$

From equation (93), one obtains:

$$\tan\zeta_m = \tanh\zeta_m \quad (96)$$

Which is termed the frequency equation for the dynamical problem, such that

$$\zeta_1 = 3.927, \quad \zeta_2 = 7.069, \quad \zeta_3 = 10.210, \ldots \quad (97)$$

Applying equations (91), (94), (95) and (97) in equations (61) and (71), one gives the displacement expression response to a moving force and a moving mass of orthotropic rectangular plate on bi-parametric elastic foundation respectively.

5. DISCUSSION OF THE ANALYTICAL SOLUTIONS

For this undamped system, it is expedient to investigate the phenomenon of resonance. So from equation (61), it is obviously shown that the orthotropic rectangular plate with elastic end conditions and on constant elastic foundation and traverse by moving distributed force with constant speed reaches a state of resonance whenever.

$$\phi_m = \Omega_{nn} \quad (98)$$

While equation (71) illustrates that the same orthotropic rectangular plate with elastic end conditions and on constant elastic foundation and traverse by moving distributed force with constant speed reaches a state of resonance when:

$$\phi_m = \omega_n \quad (99)$$

Where,

$$\omega_n = \Omega_{nn} - \frac{1}{2\Omega_{nn}}(\Omega_{nn}^2\alpha\Theta^*(\varepsilon_6 + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_7^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_8^*\frac{\sin(2k+1)\pi\varphi}{2k+1})) -k_r^2\alpha\Theta^*(\varepsilon_8 + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{23}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{24}^*\frac{\sin(2k+1)\pi\varphi}{2k+1}))) \quad (100)$$

Comparing equations (84) and (85), one obtains:

$$\vartheta_n = \Omega_{nn}[1 - \frac{1}{2\Omega_{nn}^2}(\Omega_{nn}^2\alpha\Theta^*(\varepsilon_6 + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_7^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_8^*\frac{\sin(2k+1)\pi\varphi}{2k+1})) -k_r^2\alpha\Theta^*(\varepsilon_8 + \frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{23}^*\frac{\cos(2k+1)\pi\varphi}{2k+1} - \sum_{k=1}^{\infty}\tau_{24}^*\frac{\sin(2k+1)\pi\varphi}{2k+1})))] = \Omega_{nn} \quad (101)$$

6. GRAPHS OF THE NUMERICAL SOLUTIONS

To expatiate the analysis presented in this work, orthotropic rectangular plate is assumed to be of length $L_y = 0.923m$, breadth $L_x = 0.432m$ the load velocity c = 0.8123m/s and $\varphi = 0.4m$.The results are presented on the various plotted curves below for both clamped end conditions (classical end condition) and clamped elastic end conditions (nonclassical end condition).

Figures 6.1 and 6.2 display the effect of flexural rigidity of the plate along x-axis $\boldsymbol{D_x}$ on the deflection profile of orthotropic rectangular plate with Clamped-clamped end conditions under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity $\boldsymbol{D_x}$ increases.

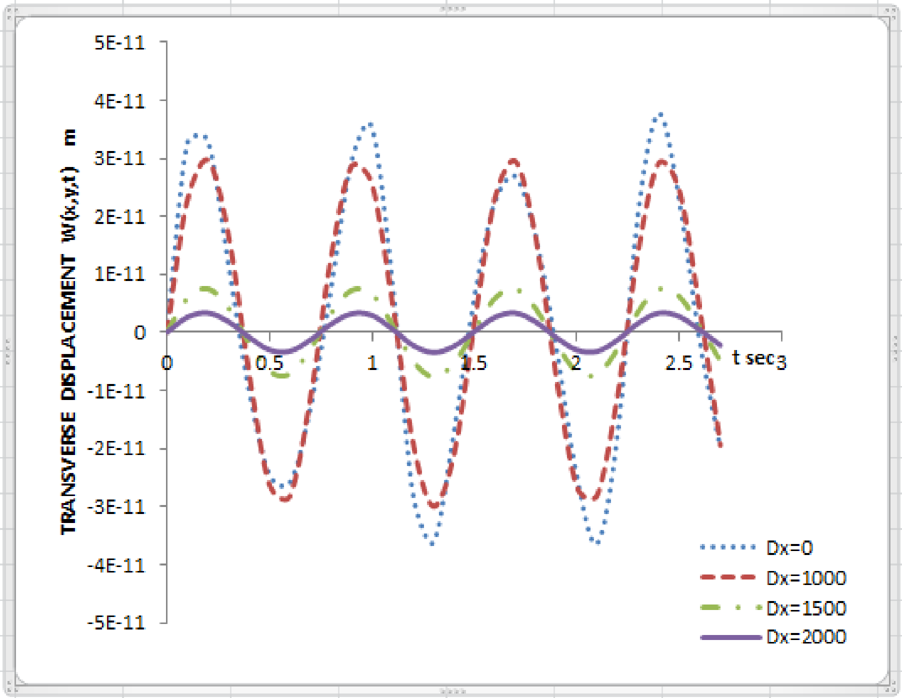

Figure 6.1: Shows displacement Profile of Orthotropic Rectangular Plate with Varying D_x with Clamped-clamped end conditions and Traversed by Moving Force

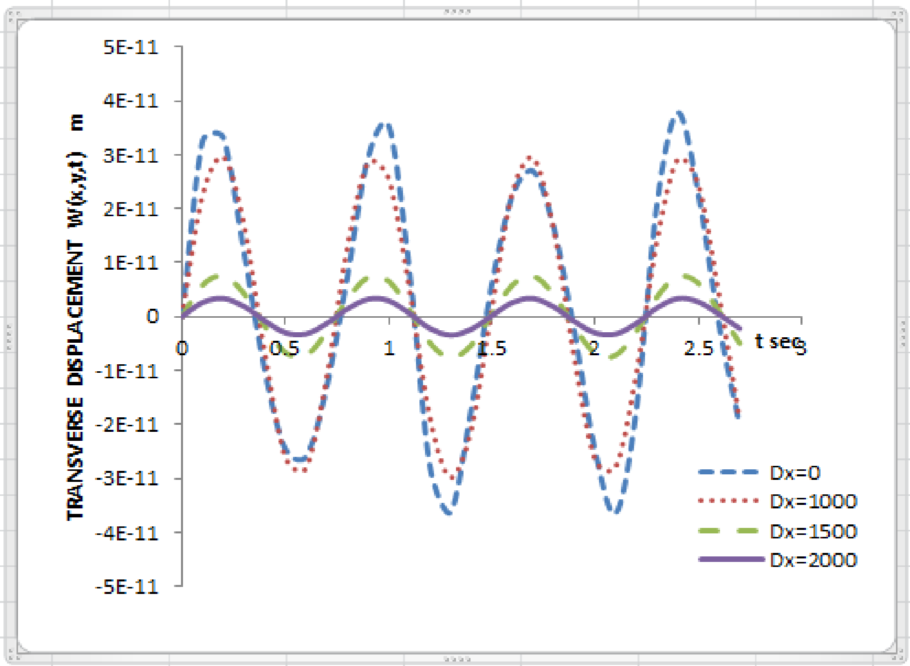

Figure 6.2: Shows displacement Profile of Orthotropic Rectangular Plate with Varying D_x with Clamped-clamped end conditions and Traversed by Moving Mass

Figures 6.3 and 6.4 display the effect of flexural rigidity of the plate along y-axis $\boldsymbol{D_y}$ on the deflection profile of orthotropic rectangular plate with Clamped-clamped end conditions under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity D_y increases.

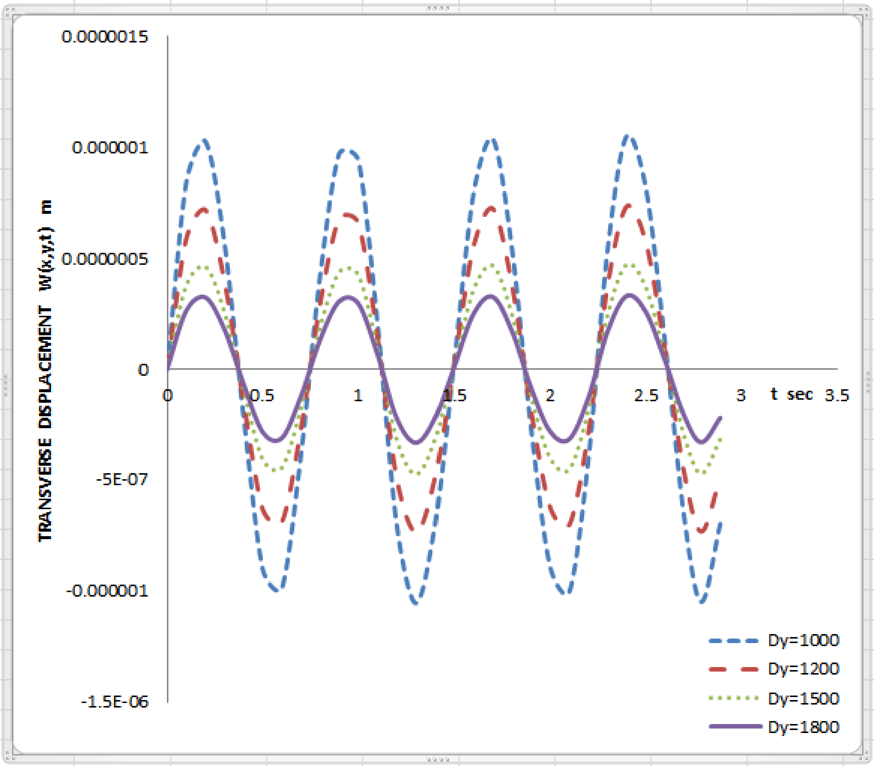

Figure 6.3: Shows displacement Profile of Orthotropic Rectangular Plate with Varying D_y with Clamped-clamped end conditions and Traversed by Moving Force

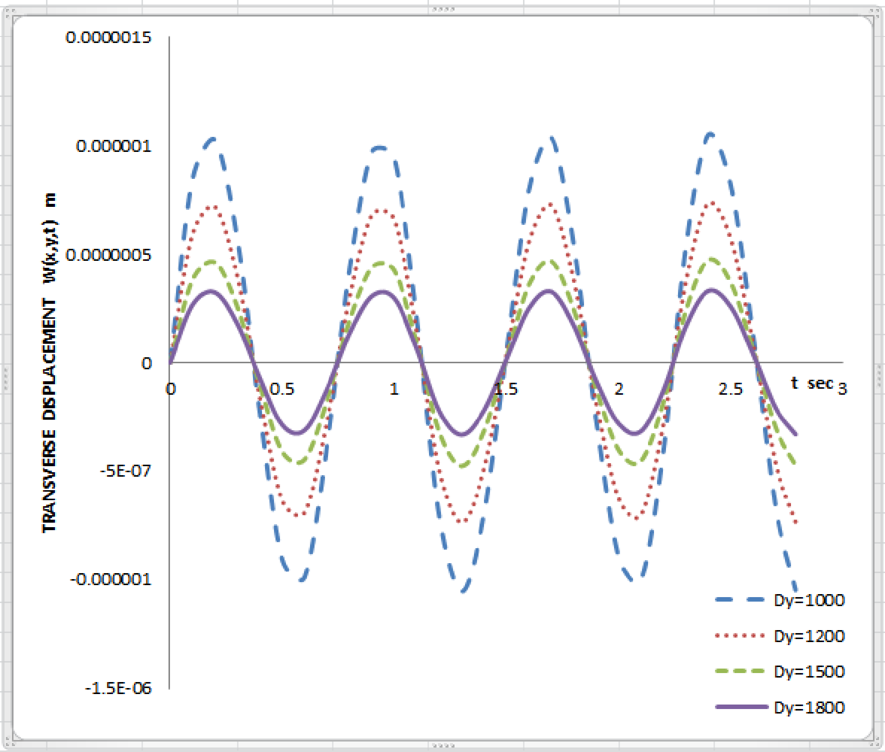

Figure 6.4: Shows displacement Profile of Orthotropic Rectangular Plate with Varying D_y with Clamped-clamped end conditions and Traversed by Moving Mass

Figures 6.5 and 6.6 display the effect of flexural rigidity of the plate along x-axis $\boldsymbol{D_x}$ on the deflection profile of orthotropic rectangular plate with Clamped elastic end conditions under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity $\boldsymbol{D_x}$ increases.

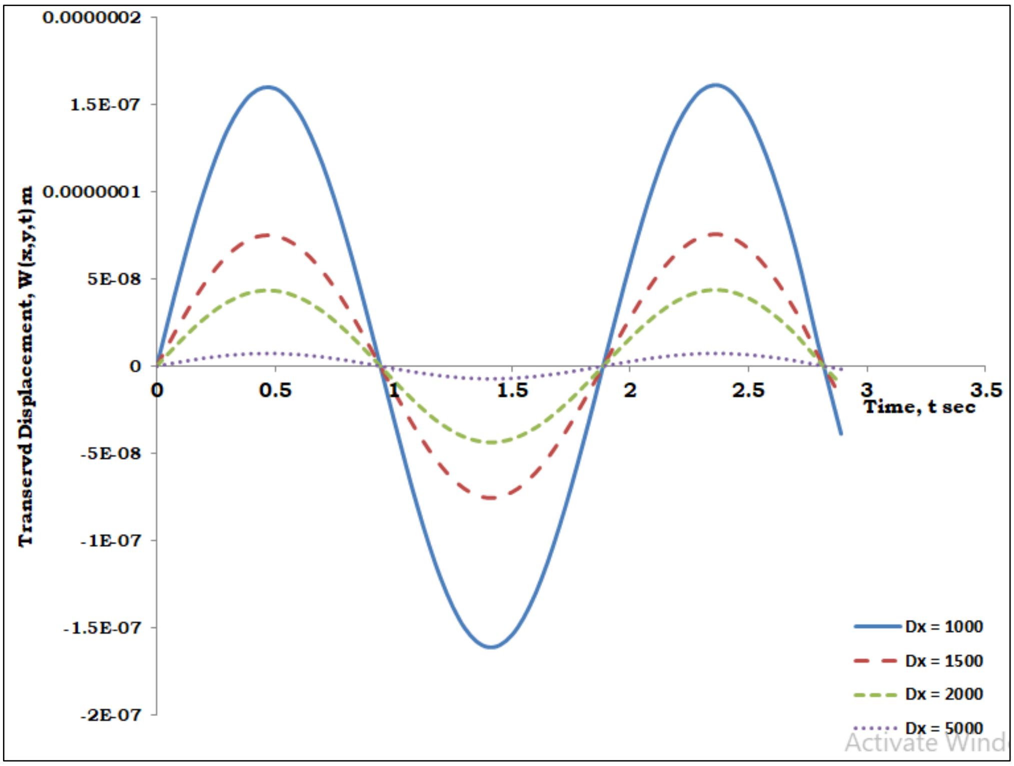

Figure 6.5: Shows displacement Profile of Orthotropic Rectangular Plate with Varying D_x with Clamped elastic end conditions and Traversed by Moving Force

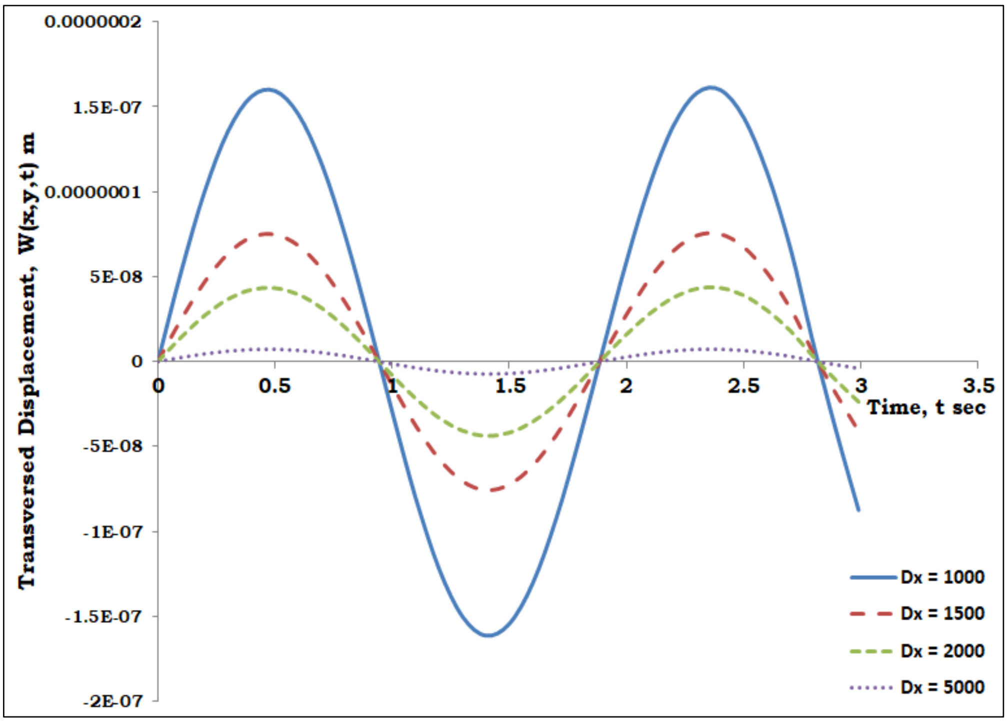

Figure 6.6: Shows displacement Profile of Orthotropic Rectangular Plate with Varying D_x with Clamped elastic end conditions and Traversed by Moving Mass

Figures 6.7 and 6.8 display the effect of flexural rigidity of the plate along y-axis D_y on the deflection profile of orthotropic rectangular plate with Clamped elastic end conditions under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity D_y increases.

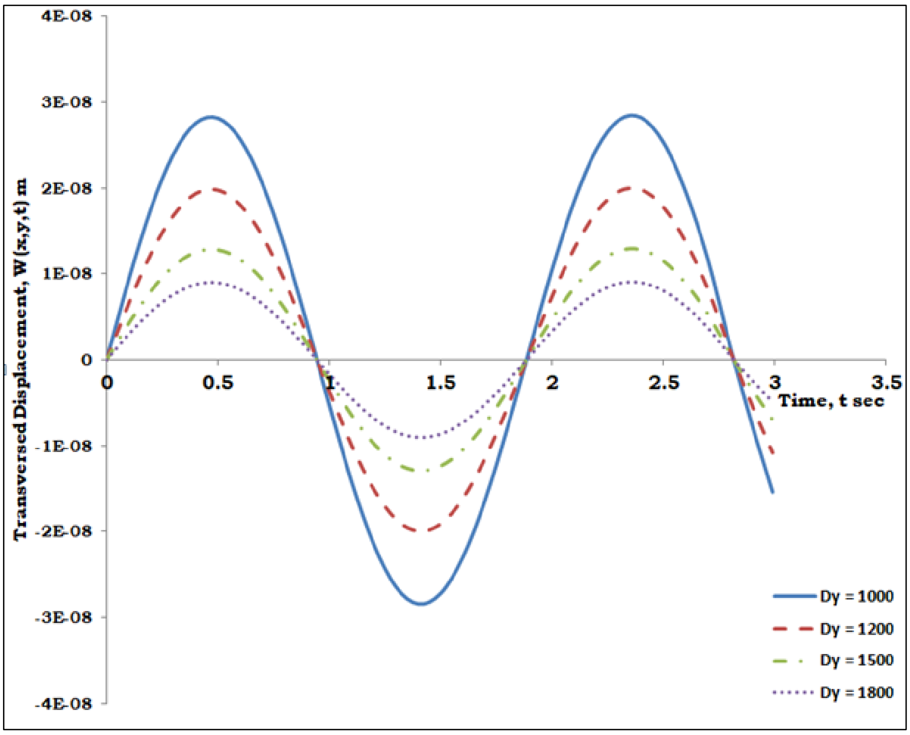

Figure 6.7: Shows displacement Profile of Orthotropic Rectangular Plate with Varying D_y with Clamped elastic end conditions and Traversed by Moving Force

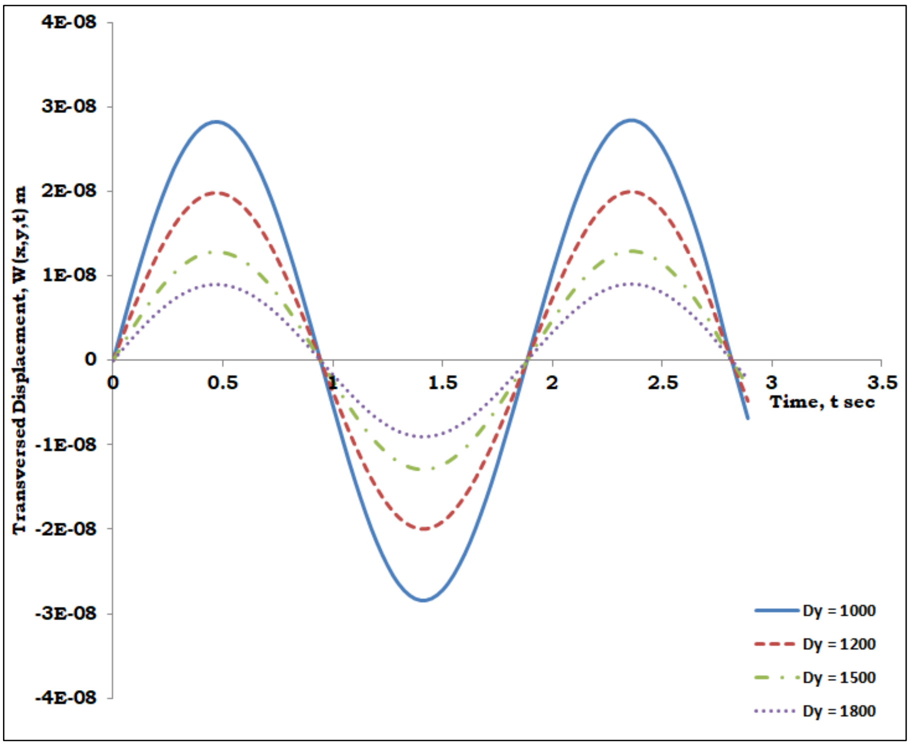

Figure 6.8: Shows displacement Profile of Orthotropic Rectangular Plate with Varying D_y with Clamped elastic end conditions and Traversed by Moving Mass

7. CONCLUSION

The dynamic influence of foundation stiffness on rectangular structural plate with constant bi-parametric elastic foundation with simple elastic end conditions has been examined in this research article. The closed form solutions to partial differential model having variable and singular coefficients of the orthotropic rectangular plates has been obtained for both cases of moving force and moving mass using the technique adopted by Shadnam *et al*., [11] which was adopted to remove the singularity in the governing fourth order partial differential equation and thereby transforming it to a sequence of second order ordinary differential equations. After making use of asymptotic technique of Struble and Laplace transformation, the analytical solution is obtained. The solutions are then interpreted. From the interpretations, it was evidence that for the equal natural frequency, the critical speed for moving mass problem is smaller than that of moving force problem. Hence resonance is achieved earlier in moving mass system than in the moving force system. The results in the plotted curves show that increase in rotatory inertia correction factor, R_o , foundation modulus K_o and shear modulus G_o resulted to decrease in the amplitudes of the orthotropic rectangular plates for both cases of moving force and moving mass problems. It is also depicted in the curves that the response amplitude of moving mass problem is higher than of moving force problem which indicates that resonance is reached earlier in moving mass problem than in moving force problem of flexural rigidity influence on dynamic response of orthotropic rectangular plate resting on constant elastic foundation.

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