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Original Research Article

Amplitude Variations of Moving Distributed Masses of Orthotropic Rectangular Plate with Elastically Supported Ends under Moving Loads

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Abstract

The deflection of thin orthotropic rectangular plate under moving loads is a classic problem in solid mechanics. However, the equations are challenging to solve due to their non linearity and complexity. At the same time, this equation is a coupled fourth order partial differential equation having variables and singular coefficients. In this research article, the partial differential equation is converted to a set of coupled second order ordinary differential equations by using a special technique adopted by Shadnam *et al.*, [19]. This transformed set of second order ordinary differential equations is then reduced using modified asymptotic method of Struble and Laplce transformation. The closed form solution is evaluated, resonance conditions are obtained and the results are showed in plotted curves to solve the variations in amplitudes for some varying orthotropic plate parameters with elastically supported ends under moving loads for both cases of moving distributed force and moving distributed mass.

Keywords: Bi-Parametric Elastic Foundation, Flexural Rigidity, Amplitude, Moving Distributed Masses, Transvered Displacement, Resonance.

1 INTRODUCTION

In the context of structural engineering and applied mechanics, an orthotropic rectangular plate simply means a plate that has different material properties in different directions. This means that the stiffness and strength of the plate can vary depending on the direction of the load. When a moving load is applied to an orthotropic rectangular plate, the plate's response can vary in terms of amplitude. The plate's deflection, stress, or strain can change as the load moves across the plate. The amplitude variation is caused by factors such as the magnitude and speed of the moving load, the material properties of the plate, and the boundary conditions of the plate. In order to analyze the amplitude variation, engineers often use mathematical models and numerical methods to calculate the plate's response at different points and time intervals as the load traverses. These analyses help in understanding how the plate behaves under the moving load and determine any potential areas of concern such as high stress concentrations or excessive deflections. As we all know an orthotropic rectangular plate has three mutually perpendicular axes: x, y and z. The x and y axes are usually aligned with the length and width of the plate, while the z axis represents the thickness. The material properties, such as stiffness and strength can vary along each of these axes. For instance, a plate may have different elastic moduli (Young's moduli) in the x, y and z directions, indicating different levels of stiffness and in the same vein, a plate may have different shear moduli and Poisson's ratios for each, representing different resistance to shear and lateral deformation, respectively. When an orthotropic rectangular structure is elastically supported on a constant elastic foundation, its dynamic response is significantly affected by flexural rigidity of the structure. The dynamic response refers to the behavior of the structure when subjected to dynamic loads or vibrations. The flexural rigidity of the structure determines the stiffness of the structure in resisting bending deformations. A higher flexural rigidity implies greater resistance to bending, resulting in a stiffer structure. Conversely, a lower flexural rigidity allows for more bending and deformations in the structure. The dynamic response of the structure is disturbed by the flexural rigidity in several ways. Firstly, a higher flexural rigidity results to a higher natural frequency of the structure. The term "natural frequency" is the frequency at which the structure vibrates without any external forces.

Many researchers in the fields of engineering, applied mathematics and mechanics have worked restlessly on plate models. Some of these researchers include: Cheung and Zinkiewicz [1], studied the problems of slabs and tanks (either isotropic or orthotropic) resting either on a semi-infinite elastic continuum or on individual springs (of the so-called Winkler's type) and solved the problems by the finite element method. Re-entrant corners, rigid walls on the slabs, concentrated moments due to bending of columns, etc., involved little computational difficulty in the method presented. Ghosh [2], studied large deflection of a rectangular plate resting on a Pasternak-type elastic foundation. The problem of large deflection of a rectangular plate resting on a Pasternak-type foundation and subjected to a uniform lateral load was being investigated by utilizing the linearized equation of plates due to H. M. Berger. The solutions derived and based on the effect of the two base parameters have been carried to practical conclusions by presenting graphs for bending moments and shear forces for a square plate with all edges simply supported.

George Z. Voyiadjis *et al.*, [3], studied thick rectangular plates on an elastic foundation.

Roknuzzama *et al.*, [4], worked on a rectangular thin plate with eccentric opening using application of finite difference method to analyze it. In their work, a rectangular steel plate subjected to uniformly distributed loading with all around simply supported edges having an eccentric rectangular opening was analyzed using the finite difference method, a straightforward numerical approach. Sobamowo *et al.*, [5], presented analytical solution of isotropic rectangular plates resting on Winkler and Pasternak foundations using Laplace transform and variation of iteration method. The dynamic analysis of isotropic thin rectangular plate resting on two-parameter elastic foundations was investigated by applying method of Laplace transform and variation of iteration method. Ibearugbulem *et al.*, [6], worked on plastic buckling analysis of thin rectangular plate under uniform in-plane compression in the longitudinal direction. They applied Taylor's series to approximate the displacement for rectangular plate clamped at all edges. Ezeh *et al*., [7], applied Galerkin's indirect variation method in analysing elastic stability of thin rectangular plate clamped at all edges. Imrak and Gerdemeli [8], presented an exact solution for thin rectangular plate. Chaurasia and Jagdish [9], calculated the large deflection and bending stresses for clamped circular plate under non-uniform load using Berge's approximate method. Silveira and Albuquerque [10], applied boundary element method to obtain the large deflection of composite laminate thin square plates clamped on the four edges. Osadebe and Aginan [11], develped Ritz mathematical and variational method and used it in the analysis of uniformly loaded clamped isotropic rectangular plate.

Werfalli and Karoud [12], researched on a free vibration analysis of rectangular plates using Galerkin based finite element method. Mama [13], investigated and even proposed a solution of free harmonic equation of simply supported plates using Galerkin-Vlasov method. Hatiegan and *et al.*, [14], examined thin clamped plates of different geometric forms using finite element method. Benamar and *et al.*, [15], examined the effects of large vibration amplitudes on the mode shapes and natural frequencies of thin isotropic plates. Alfano and Pagnotta [16], carried out a suitable approximate relationships, relating the resonance frequencies to the elastic constants of isotropic thin plates. Awodola and Adeoye [17], investigated the vibration of orthotropic rectangular plates on variable elastic Pasternak foundation with clamped end conditions. Adeoye and Awodola [118], studied the dynamic behaviour of moving distributed masses of orthotropic rectangular plate with clamped-clamped boundary conditions on constant elastic foundation.

In all the aforementioned researches, no works explicitly discussed the amplitude variation of of orthotropic rectangular plate with elastically supported ends under moving loads. In this research article, the variation in amplitudes when some varying plate parameters for orthotropic rectangular plate with elastically supported ends under moving loads are considered will be discussed explicitly.

2 Governing Equation

The transverse displacement $W(x,y,t)$ of orthotropic rectangular plates that lies on a bi-parametric elastic foundation and traversed by distributed mass M_i traversing with constant velocity $\mathrm{s_i}$ along a straight line parallel to the x-axis issuing from point y = ρ on the y-axis with flexural rigidities D_x and D_y is governed by the fourth order partial differential equation given as

$$
\begin{aligned}
&D_x \frac{\partial^4}{\partial x^4} W(x,y,t) + 2\gamma \frac{\partial^4}{\partial x^2 \partial y^2} W(x,y,t) + D_y \frac{\partial^4}{\partial y^4} W(x,y,t) + \tau \frac{\partial^2}{\partial t^2} W(x,y,t) \\
&-\vartheta h R_0 [\frac{\partial^4}{\partial x^2 \partial t^2} W(x,y,t) + \frac{\partial^4}{\partial y^2 \partial t^2} W(x,y,t)] + K_0 W(x,y,t) - G_0 [\frac{\partial^2}{\partial x^2} \\
&W(x,y,t) + \frac{\partial^2}{\partial y^2} W(x,y,t)] - \sum_{\mathrm{i}=1}^{N} [M_\mathrm{i} g H(x - s_i t) H(y-\rho) - M_\mathrm{i} (\frac{\partial^2}{\partial t^2} W(x,y,t) \\
&+2s_\mathrm{i} \frac{\partial^2}{\partial x \partial t} W(x,y,t) + \mathrm{s_i^2} \frac{\partial^2}{\partial x^2} W(x,y,t)) H(x - s_i t) H(y-\rho) W(x,y,t)] = 0
\end{aligned} \quad (1)
$$

where D_x and D_y are the flexural rigidities of the plate along x and y axes respectively.

$$D_x = \frac{\varepsilon_x h^3}{12(1-\delta_x\delta_y)}, D_y = \frac{\varepsilon_y h^3}{12(1-\delta_x\delta_y)}, B = D_x D_y + \frac{G_o h^3}{6}$$

E_x and E_y are the Young's moduli along x and y axes respectively, G_o is the rigidity modulus, δ_x and δ_y are Poisson's ratios for the material such that $\varepsilon_x\delta_y = \varepsilon_y\delta_x$, ϑ is the mass density per unit volume of the plate, h is the plate thickness, t is the time, x and y are the spatial coordinates in x and y directions respectively, R_o is the rotatory inertia correction factor, K_o is the foundation constant and g is the acceleration due to gravity, H(.) is the Heaviside function.

Re-expressing equation (2), one obtains

$$\tau\frac{\partial^2}{\partial t^2}W(x,y,t) + \tau\beta_r^2 W(x,y,t) = \vartheta h R_0[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t) + \frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)]$$
$$-2\gamma\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t) - D_x\frac{\partial^4}{\partial x^4}W(x,y,t) - D_y\frac{\partial^4}{\partial y^4}W(x,y,t) - K_0 W(x,y,t)$$
$$+\tau\kappa_n^2 W(x,y,t) + G_0[\frac{\partial^2}{\partial x^2}W(x,y,t) + \frac{\partial^2}{\partial y^2}W(x,y,t)] + \sum_{i=1}^{N}[M_i g H(x-s_i t)$$
$$H(y-\rho) - M_i(\frac{\partial^2}{\partial t^2}W(x,y,t) + 2s_i\frac{\partial^2}{\partial x\partial t}W(x,y,t) + s_i^2\frac{\partial^2}{\partial x^2}W(x,y,t))H(x-s_i t)$$
$$H(y-\rho)W(x,y,t)] \quad (2)$$

Which can be expressed further as

$$\frac{\partial^2}{\partial t^2}W(x,y,t) + \beta_r^2 W(x,y,t) = R_0[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t) + \frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)] - \frac{2\gamma}{\tau}\frac{\partial^4}{\partial x^2\partial y^2}$$
$$W(x,y,t) - \frac{D_x}{\tau}\frac{\partial^4}{\partial x^4}W(x,y,t) - \frac{D_y}{\tau}\frac{\partial^4}{\partial y^4}W(x,y,t) + [\beta_r^2 - \frac{K_0}{\mu}]W(x,y,t) + \frac{G_0}{\tau}[\frac{\partial^2}{\partial x^2}$$
$$W(x,y,t) + \frac{\partial^2}{\partial y^2}W(x,y,t)] + \sum_{i=1}^{N}[\frac{M_i g}{\tau}H(x-s_i t)H(y-\rho) - \frac{M_i}{\tau}(\frac{\partial^2}{\partial t^2}W(x,y,t) + 2s_i$$
$$\frac{\partial^2}{\partial x\partial t}W(x,y,t) + s_i^2\frac{\partial^2}{\partial x^2}W(x,y,t))H(x-s_i t)H(y-\rho)W(x,y,t)] \quad (3)$$

Where β_r^2 is the natural frequencies, $k = 1,2,3,\ldots$.
The initial conditions, without any loss of generality, is taken as

$$W(x,y,t) = 0 = \frac{\partial}{\partial t}W(x,y,t) \quad (4)$$

3 Analytical Approximate Solution

To obtain an expression for the solution of equation (4), one applies technique of Shadnam *et al.*, [11], which requires that the deflection of the plates be in series form as

$$W(x,y,t) = \sum_{r=1}^{N}\theta_r(x,y)\mu_r(t) \quad (5)$$

Where $\theta_r(x,y) = \theta_r(x)\theta_r(y)$ and

$$\theta_r(x) = \sin\frac{\gamma_r}{L_x}x + A_r\cos\frac{\gamma_r}{L_x}x + B_r\sinh\frac{\gamma_r}{L_x}x + C_r\cosh\frac{\gamma_r}{L_x}x$$

$$\theta_r(y) = \sin\frac{\gamma_r}{L_y}y + A_r\cos\frac{\gamma_r}{L_y}y + B_r\sinh\frac{\gamma_r}{L_y}y + C_r\cosh\frac{\gamma_r}{L_y}y \quad (6)$$

The right hand side of equation (4) when written in series form takes the form

$$\sum_{r=1}^{\infty}R_0[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t) + \frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)] - \frac{2\beta}{\mu}\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t) - \frac{D_x}{\tau}\frac{\partial^4}{\partial x^4}W(x,y,t)$$
$$-\frac{D_y}{\tau}\frac{\partial^4}{\partial y^4}W(x,y,t + [\beta_k^2 - \frac{K_0}{\tau}]W(x,y,t) + \frac{G_0}{\tau}[\frac{\partial^2}{\partial x^2}W(x,y,t) + \frac{\partial^2}{\partial y^2}W(x,y,t)] +$$
$$\sum_{i=1}^{N}[\frac{M_{ig}}{\tau}H(x-s_i t)H(y-\varphi) - \frac{M_i}{\tau}(\frac{\partial^2}{\partial t^2}W(x,y,t) + 2s_i\frac{\partial^2}{\partial x\partial t}W(x,y,t) + s_i^2\frac{\partial^2}{\partial x^2}$$
$$W(x,y,t))H(x-s_i t)H(y-\rho)W(x,y,t)] = \sum_{r=1}^{N}\theta_r(x,y)\alpha_r(t) \quad (7)$$

Multiplying both sides of equation (8) by $\theta_s(x,y)$, integrating on area A of the plate and considering the orthogonality of $\theta_s(x,y)$, one gets

$$\alpha_r(t) = \frac{1}{\varepsilon^*}\sum_{r=1}^{\infty}\int_A [R_0(\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t) + \frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)) - \frac{2\gamma}{\tau}\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t) - \frac{D_x}{\tau}\frac{\partial^4}{\partial x^4}W(x,y,t) - \frac{D_y}{\tau}\frac{\partial^4}{\partial y^4}W(x,y,t + (\beta_r^2 - \frac{K_0}{\mu})W(x,y,t) + \frac{G_0}{\tau}(\frac{\partial^2}{\partial x^2}W(x,y,t) + \frac{\partial^2}{\partial y^2}W(x,y,t)) + \sum_{i=1}^{N}[\frac{M_i g}{\tau}H(x-s_it)H(y-\rho) - \frac{M_i}{\tau}(\frac{\partial^2}{\partial t^2}W(x,y,t) + 2s_i\frac{\partial^2}{\partial x\partial t} W(x,y,t) + s_i^2\frac{\partial^2}{\partial x^2}W(x,y,t))H(x-s_it)H(y-\rho)W(x,y,t)]\theta_r(x,y)dA \quad (8)$$

and zero when $r \neq s$

where

$$\varepsilon^* = \int_A \theta_r^2(x,y)dA \quad (9)$$

Making use of equation (6), equation (8), taking into account equation (4), can be written as

$$\theta_r(x,y)[\ddot{\mu}_r(t) + \beta_r^2\mu_r(t)] = \frac{\theta_r(x,y)}{\varepsilon^*}\sum_{k=1}^{\infty}\int_A [R_0(\frac{\partial^2\theta_k(x,y)}{\partial x^2}\theta_s(x,y)\ddot{\mu}_k(t) + \frac{\partial^2\theta_k(x,y)}{\partial y^2} \theta_s(x,y)\ddot{\mu}_k(t)) - \frac{2\gamma}{\tau}\frac{\partial^2\theta_k(x,y)}{\partial x^2\partial y^2}\theta_s(x,y)\mu_k(t) - \frac{D_x}{\tau}\frac{\partial^4\theta_k(x,y)}{\partial x^4}\theta_s(x,y)\mu_k(t) - \frac{D_y}{\tau} \frac{\partial^4\theta_k(x,y)}{\partial y^4}\theta_s(x,y)\mu_k(t) + (\beta_k^2 - \frac{K_0}{\tau})\theta_k(x,y)\theta_s(x,y)\mu_k(t) + \frac{G_0}{\tau}(\frac{\partial^2\theta_k(x,y)}{\partial x^2}\theta_s(x,y) \mu_k(t) + \frac{\partial^2\theta_k(x,y)}{\partial y^2}\theta_s(x,y)\mu_k(t)) + \sum_{i=1}^{N}(\frac{M_i g}{\tau}\theta_s(x,y)H(x-s_it)H(y-\rho) - \frac{M_r}{\tau}(\theta_k(x,y) \theta_s(x,y)\ddot{\mu}_k(t) + 2s_i\frac{\partial\delta_q(x,y)}{\partial x}\theta_s(x,y)\dot{\mu}_k(t) + s_i^2\frac{\partial^2\delta_q(x,y)}{\partial x^2}\theta_s(x,y)\mu_k(t))H(x-s_it) H(y-\rho))]dA \quad (10)$$

When equation (10) is simplified further, one obtains

$$\ddot{\mu}_r(t) + \beta_r^2\mu_k(t) = \frac{1}{\varepsilon^*}\sum_{k=1}^{\infty}\int_A [R_0(\frac{\partial^2\theta_k(x,y)}{\partial x^2}\theta_s(x,y)\ddot{\mu}_k(t) + \frac{\partial^2\theta_k(x,y)}{\partial y^2}\theta_s(x,y)\ddot{\mu}_k(t)) - \frac{2\gamma}{\tau}\frac{\partial^2\theta_k(x,y)}{\partial x^2\partial y^2}\theta_s(x,y)\mu_k(t) - \frac{D_x}{\tau}\frac{\partial^4\theta_k(x,y)}{\partial x^4}\theta_s(x,y)\mu_k(t) - \frac{D_y}{\tau}\frac{\partial^4\theta_k(x,y)}{\partial y^4}\theta_s(x,y) \xi_q(t) + (\kappa_n^2 - \frac{K_0}{\tau})\theta_k(x,y)\theta_s(x,y)\mu_k(t) + \frac{G_0}{\tau}(\frac{\partial^2\theta_k(x,y)}{\partial x^2}\theta_s(x,y)\mu_k(t) + \frac{\partial^2\theta_k(x,y)}{\partial y^2} \theta_s(x,y)\mu_k(t)) + \sum_{i=1}^{N}(\frac{M_i g}{\tau}\theta_s(x,y)H(x-s_it)H(y-\rho) - \frac{M_i}{\tau}(\theta_k(x,y)\theta_s(x,y) \ddot{\mu}_k(t) + 2s_i\frac{\partial\theta_k(x,y)}{\partial x}\theta_s(x,y)\dot{\mu}_k(t) + s_i^2\frac{\partial^2\theta_k(x,y)}{\partial x^2}\theta_s(x,y)\mu_k(t))H(x-s_it)H(y-\rho))]dA \quad (11)$$

The system of equations in equation (11) is a set of coupled ordinary differential equations
Making use of Fourier series representation, the Heaviside functions take the form

$$H(x-s_it) = \frac{1}{4} + \frac{1}{\pi}\sum_{i=1}^{N}\frac{\sin(2r+1)\pi(x-s_it)}{2r+1}, 0 < x < 1 \quad (12)$$

$$H(y-\rho) = \frac{1}{4} + \frac{1}{\pi}\sum_{i=1}^{N}\frac{\sin(2r+1)\pi(y-\rho)}{2r+1}, 0 < y < 1 \quad (13)$$

Substituting equations (12) and (13) into equation (11) and simplifying, one obtains

$$\ddot{\mu}_r(t) + \beta_r^2\mu_r(t) - \frac{1}{\varepsilon^*}\sum_{k=1}^{\infty} [R_0\varepsilon_0\ddot{\mu}_k(t) - \frac{2\gamma}{\tau}\varepsilon_1\mu_k(t) - \frac{D_x}{\tau}\varepsilon_2\mu_k(t) - \frac{D_y}{\tau}\varepsilon_3\mu_k(t) + (\beta_r^2 - \frac{K_0}{\tau})\varepsilon_4\mu_k(t) + \frac{G_0}{\tau}\varepsilon_5\mu_k(t) - \sum_{i=1}^{N}\frac{M_i}{\tau}((\varepsilon_6 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_3^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_4^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_6^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_8^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\ddot{\mu}_k(t) + 2s_i t(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2a+1})(\sum_{f=1}^{\infty}\tau_{11}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{14}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_k(t) + s_i^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{19}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{20}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{21}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{22}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{23}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{24}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\mu_k(t))] = \sum_{k=1}^{\infty}\sum_{i=1}^{N}\frac{M_i g}{\tau\varepsilon^*}\theta_s(s_i t)\theta_s(\rho) \quad (14)$$

Which is the transformed equation that governs the problem of an orthotropic rectangular plate resting on constant bi-parametric elastic foundation.

Where

$$\tau_0 = \int_A [\frac{\partial^2}{\partial x^2}\theta_k(x,y)\theta_s(x,y) + \frac{\partial^2}{\partial y^2}\theta_k(x,y)\theta_s(x,y)]dA \quad (15)$$

$$\varepsilon_1 = \int_A \frac{\partial^2}{\partial x^2}[\frac{\partial^2}{\partial x^2}\theta_k(x,y)]\theta_s(x,y)dA \quad (16)$$

$$\varepsilon_2 = \int_A \frac{\partial^4}{\partial x^4}[\theta_k(x,y)]\theta_s(x,y)dA \quad (17)$$

$$\varepsilon_3 = \int_A \frac{\partial^4}{\partial y^4}[\theta_k(x,y)]\theta_s(x,y)dA \quad (18)$$

$$\varepsilon_4 = \int_A \theta_k(x,y)\theta_s(x,y)dA \quad (19)$$

$$\varepsilon_5 = \int_A [\frac{\partial^2}{\partial x^2}\theta_k(x,y) + \frac{\partial^2}{\partial y^2}\theta_k(x,y)]\theta_s(x,y)dA \quad (20)$$

$$\varepsilon_6 = \frac{1}{16}\int_A \delta_q(x,y)\delta_m(x,y)dA \quad (21)$$

$$\tau_1^* = \int_A \theta_k(x,y)\theta_s(x,y)\sin(2b+1)\pi x dA \quad (22)$$

$$\tau_2^* = \int_A \theta_k(x,y)\theta_s(x,y)\cos(2b+1)\pi x dA \quad (23)$$

$$\tau_3^* = \int_A \theta_k(x,y)\theta_s(x,y)\sin(2f+1)\pi y dA \quad (24)$$

$$\tau_4^* = \int_A \theta_k(x,y)\theta_s(x,y)\cos(2f+1)\pi y dA \quad (25)$$

$$\tau_5^* = \tau_1^*, \tau_6^* = \tau_2^*, \tau_7^* = \tau_3^*, \tau_8^* =_4^* \quad (26)$$

$$\varepsilon_7 = \frac{1}{16}\int_A \theta_k(x,y)\theta_s(x,y)dA \quad (27)$$

$$\tau_9^* = \int_A \frac{\partial}{\partial x}(\theta_k(x,y))\theta_s(x,y)\sin(2b+1)\pi x dA \quad (28)$$

$$\tau_{10}^* = \int_A \frac{\partial}{\partial x}(\theta_k(x,y))\theta_s(x,y)\cos(2b+1)\pi x dA \quad (29)$$

$$\tau_{11}^* = \int_A \frac{\partial}{\partial x}(\theta_k(x,y))\theta_s(x,y)\sin(2f+1)\pi y dA \quad (30)$$

$$\tau_{12}^* = \int_A \frac{\partial}{\partial x}\delta_q(x,y)\delta_m(x,y)\cos(2f+1)\pi y dA \quad (31)$$

$$\tau_{13}^* = \tau_9^*, \tau_{14}^* = \tau_{10}^*, \tau_{15}^* = \tau_{11}^*, \tau_{16}^* = \tau_{12}^* \quad (32)$$

$$\varepsilon_8 = \frac{1}{16}\int_A \frac{\partial^2}{\partial x^2}(\theta_k(x,y)\theta_s(x,y)dA \quad (33)$$

$$\tau_{17}^* = \int_A \frac{\partial^2}{\partial x^2}(\theta_k(x,y))\theta_s(x,y)\sin(2b+1)\pi x dA \quad (34)$$

$$\tau_{18}^* = \int_A \frac{\partial^2}{\partial x^2}(\theta_k(x,y))\theta_s(x,y)\cos(2b+1)\pi x dA \quad (35)$$

$$\tau_{19}^* = \int_A \theta_k(x,y)\theta_s(x,y)\sin(2f+1)\pi y dA \quad (36)$$

$$\tau_{20}^* = \int_A \frac{\partial^2}{\partial x^2}(\theta_k(x,y))\theta_s(x,y)\cos(2f+1)\pi y dA \quad (37)$$

$$\tau_{21}^* = \tau_{17}^*, \tau_{22}^* = \tau_{18}^*, \tau_{23}^* = \tau_{19}^*, \tau_{24}^* = \tau_{20}^* \quad (38)$$

$\theta_s(x,y)$ is assumed to be the products of functions $\theta_s(x)\theta_{cs}(y)$ which are the beam functions in the directions of x and y axes respectively. That is

$$\theta_s(x,y) = \theta_s(x)\theta_{cs}(y) \quad (39)$$

where

$$\theta_s(x) = \sin\lambda_s x + A_s\cos\lambda_s x + B_s\sinh\lambda_s x + C_s\cosh\lambda_s x$$
$$\theta_{cs}(y) = \sin\lambda_{cs} y + A_{cs}\cos\lambda_{cs} y + B_{cs}\sinh\lambda_{cs} y + C_{cs}\cosh\lambda_{cs} y \quad (40)$$

where $A_s, B_s, C_s, A_{cs}, B_{cs}$ and C_{cs} are constants determined by the boundary conditions while θ_s and θ_{cs} are called the mode frequencies

where

$$\lambda_s = \frac{\zeta_s}{L_x}, \lambda_{cs} = \frac{\zeta_{cs}}{L_y} \quad (41)$$

Considering a unit mass, equation (19) can be re-expressed as

$$\ddot{\mu}_r(t) + \beta_r^2\mu_r(t) - \frac{1}{\varepsilon^*}\sum_{k=1}^{\infty}\Big[R_0\varepsilon_0\ddot{\mu}_k(t) - \frac{2\gamma}{\tau}\varepsilon_1\mu_k(t) - \frac{D_x}{\tau}\varepsilon_2\mu_k(t) - \frac{D_y}{\tau}\varepsilon_3\mu_k(t)$$
$$+(\beta_k^2 - \frac{K_0}{\tau}\varepsilon_4)\mu_k(t) + \frac{G_0}{\tau}\varepsilon_5\mu_k(t) - \alpha\sigma((\varepsilon_6 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1} -$$
$$\sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_3^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_4^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*$$
$$\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_6^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\varphi}{2f+1} - \sum_{f=1}^{\infty}\tau_8^*$$

$$\frac{\sin(2f+1)\pi\rho}{2f+1}))\ddot{\mu}_k(t) + 2s_i(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}$$
$$)(\sum_{f=1}^{\infty}\tau_{11}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} -$$
$$\sum_{b=1}^{\infty}\tau_{14}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_k(t)$$

$$+s_i{}^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{19}^*$$
$$\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{20}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{21}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{22}^*$$
$$\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{23}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{24}^*\frac{\sin(2f+1)\pi\varphi}{2f+1}))\mu_k(t))\Big] \quad (42)$$
$$= \sum_{k=1}^{\infty}\frac{Mg}{\tau\varepsilon^*}\theta_s(s_i t)\theta_s(\rho)$$

equation (43) is the fundamental equation of the rectangular plate problem. where

$$\alpha = \frac{M}{\tau\sigma}, \sigma = L_x L_y \quad (43)$$
$$\theta_s(s_i t) = \sin\phi_s(t) + A_s\cos\phi_s(t) + B_s\sinh\phi_s(t) + C_s\cosh\phi_s(t) \quad (44)$$
$$\theta_s(\rho) = \sin\iota_s + A_s\cos\iota_s + B_s\sinh\iota_s + C_s\cosh\iota_s \quad (45)$$
$$\phi_s = \frac{\gamma_s s_i}{L_x}, \iota_s = \frac{\gamma_s\rho}{L_y} \quad (46)$$

3.1 Orthotropic Rectangular Plate Traversed by a Moving Force

In moving force problem in mechanics, the motion of the structure or body is being influenced by an external force that is continuously changing or moving. This force which is represented by the moving load is assumed being only transferred to the structure. In this case, the inertia effect is negligible. Setting $\alpha = 0$ in the fundamental equation (42), one obtains.

$$\ddot{\mu}_r(t) + (1 - \frac{\varepsilon_4}{\tau\varepsilon^*})\beta_r^2\mu_r(t) - \frac{1}{\tau\varepsilon^*}[\tau R_0\varepsilon_0\ddot{\mu}_r(t) - 2\beta\varepsilon_1\mu_r(t) - D_x\varepsilon_2\mu_r(t) - D_y\varepsilon_3\mu_r(t) -$$
$$K_0\varepsilon_4\mu_r(t) + G_0\varepsilon_5\mu_r(t) + \sum_{k=1,k\neq r}^{\infty}(\mu R_0\varepsilon_0\ddot{\mu}_k(t) - 2\gamma\varepsilon_1\mu_k(t) - D_x\varepsilon_2\mu_k(t) - D_y\varepsilon_3\mu_k(t) \quad (47)$$
$$+(\tau\beta_k^2 - K_0\varepsilon_4)\mu_k(t) + G_0\varepsilon_5\mu_k(t))] = \frac{Mg}{\tau\varepsilon^*}\theta_s(s_i t)\theta_s(\rho)$$

which is further be simplified as

$$\ddot{\mu}_r(t)+\Omega_r^2\mu_r(t)-\omega[\tau R_0\varepsilon_0\ddot{\mu}_r(t)-2\beta\varepsilon_1\mu_r(t)-D_x\varepsilon_2\mu_r(t)-D_y\varepsilon_3\mu_r(t)-K_0\varepsilon_4\mu_r(t) +G_0\varepsilon_5\mu_r(t)+\sum_{k=1,k\neq r}^{\infty}(\tau R_0\varepsilon_0\ddot{\mu}_k(t)-2\gamma\varepsilon_1\mu_k(t)-D_x\varepsilon_2\mu_k(t)-D_y\varepsilon_3\mu_k(t)+(\tau\beta_k^2- K_0\varepsilon_4)\mu_k(t)+G_0\varepsilon_5\mu_k(t))=\omega Mg\theta_s(s_it)\theta_s(\rho) \quad (48)$$

where $\Omega_r^2=(1-\frac{\varepsilon_4}{\tau\varepsilon^*})\beta_r^2$, $\omega=\frac{1}{\tau\varepsilon^*}$

Expanding and re-arranging equation (48), one gets

$$[1-\omega\tau R_0\varepsilon_0]\ddot{\mu}_r(t)+(\Omega_r^2-\omega\epsilon_6)\mu_r(t)-\omega\sum_{k=1,k\neq r}^{\infty}(\tau R_0\varepsilon_0\ddot{\mu}_k(t)-2\gamma\varepsilon_1\mu_k(t)-D_x\varepsilon_2 \mu_k(t)-D_y\varepsilon_3\mu_k(t)+(\tau\beta_k^2-K_0\varepsilon_4)\mu_k(t)+G_0\varepsilon_5\mu_k(t))=\omega Mg\theta_s(s_it)\theta_s(\rho) \quad (49)$$

Simplifying further, one obtains

$$\ddot{\mu}_r(t)+\frac{(\Omega_r^2-\omega J_6)}{[1-\omega\tau R_0\varepsilon_0]}\mu_r(t)+\frac{\omega}{[1-\omega\tau R_0\varepsilon_0]}\sum_{k=1,k\neq r}^{\infty}(\tau R_0\varepsilon_0\ddot{\mu}_k(t)-2\gamma\varepsilon_1\mu_k(t)-D_x\varepsilon_2 \mu_k(t)-D_y\varepsilon_3\mu_k(t)+(\tau\beta_k^2-K_0\varepsilon_4)\mu_k(t)+G_0\varepsilon_5\mu_k(t))=\frac{\omega}{[1-\omega\tau R_0\varepsilon_0]}Mg\theta_s(s_it)\theta_s(\rho) \quad (50)$$

where

$$\epsilon_6=-2\beta\varepsilon_1-D_x\varepsilon_2-D_y\varepsilon_3-K_0\varepsilon_4+G_0\varepsilon_5 \quad (51)$$

For any arbitrary ratio ω, defined as

$\omega^*=\frac{\omega}{1+\omega}$, one obtains

$$\omega=\frac{\omega^*}{1-\omega^*}=\omega^*+o(\omega^{*2})+\ldots$$

For only $o(\omega^*)$, one obtains

$\omega=\omega^*$

On application of binomial expansion,

$$\frac{1}{1-\omega^*\tau R_0\varepsilon_0}=1+\omega^*\tau R_0\sigma_0+o(\omega^{*2})+\ldots \quad (52)$$

On putting equation (52) into equation (50), one obtains

$$\ddot{\mu}_r(t)+(\Omega_r^2-\omega^*\epsilon_6)(1+\omega^*\mu R_0\varepsilon_0+o(\omega^{*2})+\ldots)\mu_r(t)+\omega^*(1+\omega^*\tau R_0\varepsilon_0+o(\omega^{*2}) +\ldots)\sum_{k=1,k\neq r}^{\infty}(\tau R_0\varepsilon_0\ddot{\mu}_k(t)-2\gamma\varepsilon_1\mu_k(t)-D_x\varepsilon_2\mu_k(t)-D_y\varepsilon_3\mu_k(t)+(\tau\beta_k^2-K_0\varepsilon_4) \mu_k(t)+G_0\varepsilon_5\mu_k(t))=\omega^*Mg(1+\omega^*\tau R_0\sigma_0+o(\omega^{*2})+\ldots)\theta_s(s_it)\theta_s(\rho) \quad (53)$$

Retaining only $o(\Psi^*)$, equation (54) becomes

$$\ddot{\mu}_r(t)+[\Omega_r^2(1+\omega^*\tau R_0\sigma_0)-\omega^*\epsilon_6]\mu_r(t)+\omega^*\sum_{k=1,k\neq r}^{\infty}(\tau R_0\varepsilon_0\ddot{\mu}_k(t)-2\gamma\varepsilon_1\mu_k(t)-D_x\varepsilon_2 \mu_k(t)-D_y\varepsilon_3\mu_k(t)+(\tau\beta_k^2-K_0\varepsilon_4)\mu_k(t)+G_0\varepsilon_5\xi_q(t))=\omega^*Mg\theta_s(s_it)\theta_s(\rho) \quad (54)$$

which is simplified further as

$$\ddot{\mu}_r(t)+[\Omega_r^2(1+\omega^*\tau R_0\sigma_0)-\omega^*\epsilon_6]\mu_r(t)+\omega^*\sum_{k=1,k\neq r}^{\infty}[\tau R_0\varepsilon_0\ddot{\mu}_k(t)-[2\gamma\varepsilon_1 -D_x\varepsilon_2-D_y\varepsilon_3-(\tau\beta_k^2-K_0\varepsilon_4)+G_0\varepsilon_5]\mu_k(t)]=\omega^*Mg\theta_s(s_it)\theta_s(\rho) \quad (55)$$

Using Struble's technique, one obtains

$$\Omega_{rr}=\Omega_r-(\frac{\Omega_r^2-\epsilon_7}{2\Omega_r}) \quad (56)$$

Represents the modified frequency for moving force problem.
where

$$\epsilon_7=[\Omega_r^2(1+\omega^*\tau R_0\varepsilon_0)-\Psi^*\epsilon_6] \quad (57)$$

Using equation (58), the homogeneous part of equation (55) can be written as

$$\ddot{\mu}_r(t)+\Omega_{rr}^2\mu_r(t)=0 \quad (58)$$

Hence, the entire equation (56) gives

$$\ddot{\mu}_r(t) + \Omega_{rr}^2\mu_r(t) = \omega^* Mg\theta_s(s_i t)\theta_s(\rho) \tag{59}$$

On solving equation (59) one obtains

$$\mu_r(t) = \frac{Mg\omega^*\theta_s(\rho)}{\Omega_{rr}(\phi_s^4-\Omega_{rr}^4)}[(\phi_s^2+\Omega_{rr}^2)(\phi_s\sin\Omega_{rr}t - \Omega_{rr}\sin\phi_s t) - A_s\Omega_{rr}(\phi_s^2+\Omega_{rr}^2)(\cos\phi_s t - \cos\Omega_{rr}t) - B_s(\phi_s^2-\Omega_{rr}^2)(\phi_s\sin\Omega_{rr}t - \Omega_{rr}\sinh\phi_s t) + C_s\Omega_{rr}(\phi_s^2 - \Omega_{rr}^2)(\cosh\phi_s t - \cos\Omega_{rr}t)] \tag{60}$$

Making use of equation (5) one obtains

$$W(x,y,t) = \sum_{s=1}^{\infty}\sum_{is=1}^{\infty}\frac{Mg\omega^*\theta_s(\rho)}{\Omega_{rr}(\phi_s^4-\Omega_{rr}^4)}[(\phi_s^2+\Omega_{rr}^2)(\phi_s\sin\Omega_{rr}t - \Omega_{rr}\sin\phi_s t) - A_s\Omega_{rr}(\phi_s^2+\Omega_{rr}^2)(\cos\phi_s t - \cos\Omega_{rr}t) - B_s(\phi_s^2-\Omega_{rr}^2)(\phi_s\sin\Omega_{rr}t - \Omega_{rr}\sinh\phi_s t) + C_s\Omega_{rr}(\phi_s^2-\Omega_{rr}^2)(\cosh\phi_s t - \cos\Omega_{rr}t)](\sin\frac{\zeta_s}{L_x}x + A_s\cos\frac{\zeta_s}{L_x}x + B_s\sinh\frac{\zeta_s}{L_x}x + C_s\cosh\frac{\zeta_s}{L_x}x)(\sin\frac{\zeta_{is}}{L_y}y + A_{is}\cos\frac{\zeta_{is}}{L_y}y + B_{is}\sinh\frac{\zeta_{is}}{L_y}y + C_{is}\cosh\frac{\zeta_{is}}{L_y}y) \tag{61}$$

Which stands for the transverse displacement response to a moving force problem of orthotropic rectangular plate.

3.2 Orthtropic Rectangular Plate Traversed by a Moving Mass

In moving mass problem, the system or body is subjected to an external force or forces as it moves. The behaviour of the system is influenced by the interaction occurred between the applied forces and the system's mass, which results in various changes, effects and phenomena. That is to say, the weight and as well as inertia forces are transferred to the moving load. That is the inertia effect is not negligible. That is, $\alpha \neq 0$ and so it is expedient to solve the entire equation [42].

To solve this equation,one make use of an analytical approximate method. This method is known as an approximate analytical method of Struble. The homogeneous part of equation [42], shall be replaced by a free system operator defined by the modified frequency Ω_{rr}. Thus, the entire equation becomes.

$$\ddot{\mu}_r(t) + \beta_r^2\mu_r(t) - \frac{1}{\varepsilon^*}\sum_{k=1}^{\infty}[R_0\varepsilon_0\ddot{\mu}_k(t) - \frac{2\gamma}{\tau}\varepsilon_1\mu_k(t) - \frac{D_x}{\tau}\varepsilon_2\mu_k(t) - \frac{D_y}{\tau}\varepsilon_3\mu_k(t) + (\beta_k^2 - \frac{K_0}{\tau}\varepsilon_4)\mu_k(t) + \frac{G_0}{\tau}\varepsilon_5\mu_k(t) - \alpha\sigma((\varepsilon_6 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_3^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_4^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_6^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\varphi}{2f+1} - \sum_{f=1}^{\infty}\tau_8^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\ddot{\mu}_k(t) + 2s_i(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{11}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{14}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_k(t) \tag{62}$$

$$+ {s_i}^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{19}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{20}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{21}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{22}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{23}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{24}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\mu_k(t))]$$

$$= \sum_{k=1}^{\infty}\frac{Mg}{\tau\varepsilon^*}\theta_s(s_i t)\theta_s(\rho)$$

where $\vartheta^* = \frac{\sigma}{\varepsilon^*}$

On expanding and simplifying equation (62), one obtains

$$\ddot{\mu}_r(t) + \beta_r^2\mu_r(t) + \alpha\vartheta^*[(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) (\sum_{f=1}^{\infty}\tau_3^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_4^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_6^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_8^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\ddot{\mu}_r(t) + 2s_i(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2a+1} - \sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}$$

$$)(\sum_{f=1}^{\infty}\tau_{11}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{14}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_k(t) + {s_i}^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) (\sum_{f=1}^{\infty}\tau_{19}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{20}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{21}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{23}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{24}^*\frac{\sin(2f+1)\pi\varphi}{2f+1}$$

$$))\mu_r(t)] + \alpha\vartheta^*\sum_{k=1,k\neq r}^{\infty}[(\varepsilon_6 + \frac{1}{\pi^2}(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_3^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_4^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_6^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_8^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\ddot{\mu}_k(t) + 2s_i(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2a+1} - \sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{11}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{14}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_k(t) + {s_i}^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{19}^*\frac{\cos(2f+1)\pi\rho}{2f+1} -$$

$$\sum_{f=1}^{\infty}\tau_{20}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{21}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{22}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{23}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{24}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\mu_k(t)] = \omega^* Mg\theta_s(s_i t)\theta_s(\rho) \quad (63)$$

On further rearrangements and simplifications, one obtains

$$(1+\alpha\vartheta^*(\varepsilon_6+\frac{1}{\pi^2}(\varepsilon_6+\frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-\sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_3^*$$

$$\frac{\cos(2f+1)\pi\rho}{2f+1}-\sum_{f=1}^{\infty}\tau_4^*\frac{\sin(2f+1)\pi\rho}{2f+1})+\frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-\sum_{b=1}^{\infty}\tau_6^*\frac{\sin(2b+1)\pi s_i t}{2b+1}$$

$$)+\frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\rho}{2f+1}-\sum_{f=1}^{\infty}\tau_8^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\ddot{\mu}_k(t)+2s_i\alpha\vartheta^*(\varepsilon_7+\frac{1}{\pi^2}$$

$$(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2a+1}-\sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{11}^*\frac{\cos(2f+1)\pi\rho}{2f+1}-$$

$$\sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1})+\frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-\sum_{b=1}^{\infty}\tau_{14}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}$$

$$)+\frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1}-\sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_r(t)+(\Omega_{rr}^2+\alpha\vartheta^* s_i^2(\varepsilon_8$$

$$+\frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-\sum_{b=1}^{\infty}\tau_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{19}^*\frac{\cos(2f+1)\pi\rho}{2f+1}$$

$$-\sum_{k=1}^{\infty}\tau_{20}^*\frac{\sin(2k+1)\pi\varphi}{2k+1})+\frac{1}{4\pi}(\sum_{a=1}^{\infty}\tau_{21}^*\frac{\cos(2a+1)\pi k_r t}{2a+1}-\sum_{a=1}^{\infty}\tau_{22}^*\frac{\sin(2a+1)\pi k_r t}{2a+1}$$

$$)+\frac{1}{4\pi}(\sum_{k=1}^{\infty}\tau_{23}^*\frac{\cos(2k+1)\pi\varphi}{2k+1}-\sum_{k=1}^{\infty}\tau_{24}^*\frac{\sin(2k+1)\pi\varphi}{2k+1})))\mu_r(t)+\alpha\vartheta^*$$

$$\alpha\vartheta^*\sum_{k=1,k\neq r}^{\infty}[(\varepsilon_6+\frac{1}{\pi^2}(\varepsilon_6+\frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-\sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1})$$

$$(\sum_{f=1}^{\infty}\tau_3^*\frac{\cos(2f+1)\pi\rho}{2f+1}-\sum_{f=1}^{\infty}\tau_4^*\frac{\sin(2f+1)\pi\rho}{2f+1})+\frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-$$

$$\sum_{b=1}^{\infty}\tau_6^*\frac{\sin(2b+1)\pi s_i t}{2b+1})+\frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\rho}{2f+1}-\sum_{f=1}^{\infty}\tau_8^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\ddot{\mu}_k(t)$$

$$+2s_i(\varepsilon_7+\frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2a+1}-\sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{11}^*\frac{\cos(2f+1)\pi\rho}{2f+1}$$

$$-\sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1})+\frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-\sum_{b=1}^{\infty}\tau_{14}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})$$

$$+\frac{1}{4\pi}(\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1}-\sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_k(t)+{s_i}^2(\varepsilon_8+$$

$$\frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-\sum_{b=1}^{\infty}\tau_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{19}^*\frac{\cos(2f+1)\pi\rho}{2f+1}-$$

$$\sum_{f=1}^{\infty}\tau_{20}^*\frac{\sin(2b+1)\pi\rho}{2b+1})+\frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{21}^*\frac{\cos(2b+1)\pi s_i t}{2b+1}-\sum_{a=1}^{\infty}\tau_{22}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})+\frac{1}{4\pi}$$
$$(\sum_{f=1}^{\infty}\tau_{23}^*\frac{\cos(2f+1)\pi\rho}{2f+1}-\sum_{b=1}^{\infty}\tau_{24}^*\frac{\sin(2b+1)\pi\rho}{2b+1}))\mu_k(t)]=\omega^* Mg\theta_s(s_i t)\theta_s(\rho) \qquad (64)$$

Further expression of equation (65) gives

$$\ddot{\mu}_r(t) + 2s_i\alpha\vartheta^*(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2a+1} - \sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{11}^* \frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{14}^* \frac{\sin(2b+1)\pi s_i t}{2b+1}) + (\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_k(t) +$$

$$(\Omega_{rr}^2(1 - \alpha\vartheta^*(\varepsilon_6 + \frac{1}{\pi^2}(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) (\sum_{f=1}^{\infty}\tau_3^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_4^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_6^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + (\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_8^*\frac{\sin(2f+1)\pi\rho}{2f+1})))) +$$

$$s_i^2\alpha\vartheta^*(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{b=1}^{\infty}\tau_{19}^* \frac{\cos(2b+1)\pi\rho}{2b+1} - \sum_{b=1}^{\infty}\tau_{20}^*\frac{\sin(2b+1)\pi\rho}{2b+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{21}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{22}^* \frac{\sin(2b+1)\pi s_i t}{2b+1}) + (\sum_{b=1}^{\infty}\tau_{23}^*\frac{\cos(2b+1)\pi\rho}{2b+1} - \sum_{b=1}^{\infty}\tau_{24}^*\frac{\sin(2b+1)\pi\rho}{2b+1}))\mu_r(t)$$

$$+\alpha\vartheta^*\sum_{k=1,k\neq r}^{\infty}[(\varepsilon_6 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_1^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_2^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{b=1}^{\infty} \tau_3^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{b=1}^{\infty}\tau_4^*\frac{\sin(2b+1)\pi\rho}{2b+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_5^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_6^* \frac{\sin(2b+1)\pi s_i t}{2b+1}) + (\sum_{f=1}^{\infty}\tau_7^*\frac{\cos(2f+1)\pi\rho}{2k+1} - \sum_{f=1}^{\infty}\tau_8^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\ddot{\mu}_k(t) +$$

$$2s_i(\varepsilon_7 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_9^*\frac{\cos(2b+1)\pi s_i t}{2a+1} - \sum_{b=1}^{\infty}\tau_{10}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{11}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{12}^*\frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{13}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{14}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + (\sum_{f=1}^{\infty}\tau_{15}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{16}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\dot{\mu}_k(t) + s_i^2(\varepsilon_8 + \frac{1}{\pi^2}(\sum_{b=1}^{\infty}\tau_{17}^* \frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}E_{18}^*\frac{\sin(2b+1)\pi s_i t}{2b+1})(\sum_{f=1}^{\infty}\tau_{19}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{20}^* \frac{\sin(2f+1)\pi\rho}{2f+1}) + \frac{1}{4\pi}(\sum_{b=1}^{\infty}\tau_{21}^*\frac{\cos(2b+1)\pi s_i t}{2b+1} - \sum_{b=1}^{\infty}\tau_{22}^*\frac{\sin(2b+1)\pi s_i t}{2b+1}) + \frac{1}{4\pi} (\sum_{f=1}^{\infty}\tau_{23}^*\frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty}\tau_{24}^*\frac{\sin(2f+1)\pi\rho}{2f+1}))\mu_k(t)] = \omega^* Mg\theta_s(s_i t)\theta_s(\rho) \quad (65)$$

Applying the modified asymptotic method of Struble, equation (65) can be re-expressed as

$\ddot{\mu}_r(t) + \epsilon_r^2\mu_r(t) = 0$ (66)

for the homogeneous case

Hence, the entire equation becomes

$$\ddot{\mu}_r(t) + \epsilon_r^2\mu_r(t) = \omega^* Mg\theta_s(s_i t)\theta_s(\rho) \quad (67)$$

Where

$$\epsilon_{\mathrm{r}} = \Omega_{\mathrm{rr}} - \frac{1}{2\Omega_{\mathrm{rr}}}[\Omega_{\mathrm{rr}}^2 \alpha\Theta^*(\varepsilon_6 + \frac{1}{4\pi}(\sum_{\mathrm{f=1}}^{\infty} \tau_7^* \frac{\cos(2\mathrm{f}+1)\pi\rho}{2\mathrm{f}+1} - \sum_{\mathrm{f=1}}^{\infty} \tau_8^* \frac{\sin(2\mathrm{f}+1)\pi\rho}{2\mathrm{f}+1})) \\ -\mathrm{s_i^2}\alpha\vartheta^*(\varepsilon_8 + \frac{1}{4\pi}(\sum_{\mathrm{f=1}}^{\infty} \tau_{23}^* \frac{\cos(2\mathrm{f}+1)\pi\rho}{2\mathrm{f}+1} - \sum_{\mathrm{f=1}}^{\infty} \tau_{24}^* \frac{\sin(2\mathrm{f}+1)\pi\rho}{2\mathrm{f}+1}))] \quad (68)$$

Which gives the modified frequency representing the frequency of the free system.
Rewriting equation (67), one obtains

$$\ddot{\mu}_{\mathrm{r}}(t) + \epsilon_{\mathrm{r}}^2\mu_{\mathrm{r}}(t) = \omega^*\mathrm{Mg}\theta_{\mathrm{s}}(\rho)[\sin\phi_s(t) + A_s\cos\phi_s t + B_s\sinh\phi_s t + C_s\cosh\phi_s t] \quad (69)$$

Making use of the procedures applied to solve equation (59) earlier, one obtains

$$\epsilon_{\mathrm{r}}(t) = \frac{\omega^* Mg\theta_{\mathrm{s}}(\rho)}{\epsilon_{\mathrm{r}}(\phi_s^4 - \epsilon_{\mathrm{r}}^4)}[(\phi_s^2 + \epsilon_{\mathrm{r}}^2)(\phi_s\sin\epsilon_{\mathrm{r}}t - \epsilon_{\mathrm{r}}\sin\phi_s t) - A_s\epsilon_{\mathrm{r}}(\phi_s^2 + \epsilon_{\mathrm{r}}^2)(\cos\phi_s t - \\ \cos\epsilon_{\mathrm{r}}t) - B_s(\phi_s^2 - \epsilon_{\mathrm{r}}^2)(\phi_s\sin\epsilon_{\mathrm{r}}t - \epsilon_{\mathrm{r}}\sinh\phi_s t) + C_s\epsilon_{\mathrm{r}}(\phi_s^2 - \epsilon_{\mathrm{r}}^2)(\cosh\phi_s t \\ -\cos\epsilon_{\mathrm{r}}t)] \quad (70)$$

In reference to equation (5), one gets

$$W(x,y,t) = \sum_{\mathrm{s=1}}^{\infty}\sum_{\mathrm{is=1}}^{\infty} \frac{\omega^* Mg\theta_{\mathrm{s}}(\rho)}{\epsilon_{\mathrm{r}}(\phi_s^4 - \epsilon_{\mathrm{r}}^4)}[(\phi_s^2 + \epsilon_{\mathrm{r}}^2)(\phi_s\sin\epsilon_{\mathrm{r}}t - \epsilon_{\mathrm{r}}\sin\phi_s t) - A_s\epsilon_{\mathrm{r}}(\phi_s^2 \\ +\epsilon_{\mathrm{r}}^2)(\cos\phi_s t - \cos\epsilon_{\mathrm{r}}t) - B_s(\phi_s^2 - \epsilon_{\mathrm{r}}^2)(\phi_s\sin\epsilon_{\mathrm{r}}t - \epsilon_{\mathrm{r}}\sinh\phi_s t) + C_s\epsilon_{\mathrm{r}}(\phi_s^2 \\ -\epsilon_{\mathrm{r}}^2)(\cosh\phi_s t - \cos\epsilon_{\mathrm{r}}t)](\sin\frac{\zeta_s}{L_x}x + A_s\cos\frac{\zeta_s}{L_x}x + B_s\sinh\frac{\zeta_s}{L_x}x + C_s \\ \cosh\frac{\zeta_s}{L_x}x)(\sin\frac{\zeta_{\mathrm{is}}}{L_y}y + A_{\mathrm{is}}\cos\frac{\zeta_{\mathrm{is}}}{L_y}y + B_{\mathrm{is}}\sinh\frac{\zeta_{\mathrm{is}}}{L_y}y + C_{\mathrm{is}}\cosh\frac{\zeta_{\mathrm{is}}}{L_y}y) \quad (71)$$

which is the transverse displacement response to a moving mass of an orthotropic rectangular plate.

4 Illustrative Examples

Orthotropic Rectangular Plate with Elastic Elastic Boundary Conditions

For the case when the orthotropic plate is elastically supported both at $x = 0$ and $x = L_x$ and also at $y = 0$ and $y = L_y$, the conditions take are the form

$$W''(0,L_y,t) - \varphi_1 W'(0,L_y,t) = 0 = W'''(0,L_y,t) + \varphi_2 W(0,L_y,t) \quad (72)$$

at the end $x = 0$ and

$$W''(L_x,L_y,t) - \varphi_1 W'(L_x,L_y,t) = 0 = W'''(L_x,L_y,t) + \varphi_2 W(L_x,L_y,t) \quad (73)$$

at the end $x = L_x$

In the same, we have

$$W''(0,L_y,t) - \varphi_1 W'(0,L_y,t) = 0 = W'''(0,L_y,t) + \varphi_2 W(0,L_y,t) \quad (74)$$

at the end $y = 0$ and

$$W''(L_x,L_y,t) - \varphi_1 W'(L_x,L_y,t) = 0 = W'''(L_x,L_y,t) + \varphi_2 W(L_x,L_y,t) \quad (75)$$

at the end $y = L_y$

Thus, for normal modes, we have

$$\zeta_s''(0) - \varphi_1\zeta_s'(0) = 0 = \zeta_s'''(0) + \varphi_2\zeta_s(0) \\ \zeta_{\mathrm{is}}''(0) - \varphi_1\zeta_{\mathrm{is}}'(0) = 0 = \zeta_{\mathrm{is}}'''(0) + \varphi_2\zeta_{\mathrm{is}}(0) \quad (76)$$

at the end $x = 0$ and $y = 0$ and

$$\zeta_s''(L_x) - \varphi_1\zeta_s'(L_x) = 0 = \zeta_s'''(L_x) + \varphi_2\zeta_s(L_x) \\ \zeta_{\mathrm{is}}''(L_y) - \varphi_1\zeta_{\mathrm{is}}'(L_y) = 0 = \zeta_{\mathrm{is}}'''(L_y) + \varphi_2\zeta_{\mathrm{is}}(L_y) \quad (77)$$

Using equations (87) and (88), it can be shown that

$$C_s = \frac{\left[\frac{\zeta_s}{L_x} - k_1r_2\right]\sin\zeta_s + \left[\frac{r_2\zeta_s}{L_x} + k_1\right]\cosh\zeta_s - \frac{r_1\zeta_s}{L_x}\sinh\zeta_s + k_1r_1\cosh\zeta_s}{k_1r_1\sin\zeta_s - \frac{r_1\zeta_s}{L_x}\cos\zeta_s + \left[\frac{r_3\zeta_s}{L^3_x} - k_1\right]\sinh\zeta_s + \left[\frac{\zeta_s}{L_x} - k_1r_3\right]\cosh\zeta_s} \\ = \frac{-\left[\frac{r_2\zeta^3{}_s}{L^3{}_x} + k_2\right]\sin\zeta_s + \left[\frac{\zeta^3{}_s}{L^3{}_x} - k_2r_2\right]\cos\zeta_s - k_2r_1\sinh\zeta_s - \frac{r_1\zeta^3{}_s}{L^3{}_x}\cosh\zeta_s}{\frac{r_1\zeta^3{}_s}{L^3{}_x}\sin\zeta_s + k_2r_1\cos\zeta_s + \left[\frac{\zeta^3{}_s}{L^3{}_x} + k_2r_3\right]\sinh\zeta_s + \left[\frac{r_3\zeta_s}{L_x} + k_2\right]\cosh\zeta_s} \quad (78)$$

$A_s = r_1 C_s + r_2\, , B_s = r_3 C_s + r_1,$ (79)

where

$$r_1 = \frac{\frac{\zeta^4{}_s}{L^4{}_x} + k_1 k_2}{\frac{\zeta^4{}_s}{L^4{}_x} - k_1 k_2}\, , r_2 = -\frac{\frac{2k_1 \zeta^3{}_s}{L^3{}_x}}{\frac{\zeta^4{}_s}{L^4{}_x} - k_1 k_2}\, , r_3 = -\frac{\frac{2k_1 \zeta_s}{L_x}}{\frac{\zeta^4{}_s}{L^4{}_x} - k_1 k_2} \quad (80)$$

Similarly, we have

$$C_{is} = \frac{\left[\frac{\zeta_{is}}{L_x} - k_1 r_2\right] \sin \zeta_{is} + \left[\frac{r_2 \zeta_{is}}{L_y} + k_1\right] \cosh \zeta_{is} - \frac{r_1 \zeta_s}{L_y} \sinh \zeta_{is} + k_1 r_1 \cosh \zeta_{is}}{k_1 r_1 \sin \zeta_{is} - \frac{r_1 \zeta_{is}}{L_y} \cos \zeta_{is} + \left[\frac{r_3 \zeta_{is}}{L^3{}_y} - k_1\right] \sinh \zeta_{is} + \left[\frac{\zeta_{is}}{L_y} - k_1 r_3\right] \cosh \zeta_{is}}$$

$$= \frac{-\left[\frac{r_2 \zeta^3{}_{is}}{L^3{}_y} + k_2\right] \sin \zeta_{is} + \left[\frac{\zeta^3{}_{is}}{L^3{}_y} - k_2 r_2\right] \cos \zeta_{is} - k_2 r_1 \sinh \zeta_{is} - \frac{r_1 \zeta^3{}_{is}}{L^3{}_y} \cosh \zeta_{is}}{\frac{r_1 \zeta^3{}_{is}}{L^3{}_y} \sin \zeta_{is} + k_2 r_1 \cos \zeta_{is} + \left[\frac{\zeta^3{}_{is}}{L^3{}_y} + k_2 r_3\right] \sinh \zeta_{is} + \left[\frac{r_3 \zeta_{is}}{L_y} + k_2\right] \cosh \zeta_{is}} \quad (81)$$

$A_{is} = r_1 C_{is} + r_2\, , B_{is} = r_3 C_{is} + r_1,$ (82)

Where

$$r_1 = \frac{\frac{\zeta^4{}_{is}}{L^4{}_y} + k_1 k_2}{\frac{\zeta^4{}_{is}}{L^4{}_y} - k_1 k_2}\, , r_2 = -\frac{\frac{2k_1 \zeta^3{}_{is}}{L^3{}_y}}{\frac{\zeta^4{}_{is}}{L^4{}_y} - k_1 k_2}\, , r_3 = -\frac{\frac{2k_1 \zeta_{is}}{L_x}}{\frac{\zeta^4{}_{is}}{L^4{}_y} - k_1 k_2} \quad (83)$$

From equations (78), (79) and (80), one obtains the frequency equation for the dynamical problem is obtained as

$\tan\zeta_s = \tanh\zeta_s$ (84)

Hence, we have

$\zeta_1 = 3.927, \zeta_2 = 7.069, \zeta_3 = 10.210, \ldots$ (85)

Applying equations (78), (79) and (85) in equations (61) and (71), one gives the displacement expression response to a moving force and a moving mass of orthotropic rectangular plate on bi-parametric elastic foundation respectively.

5 Discusion of the Analytical Solutions

For this undamped system, it is necessary to investigate the phenomenon of resonance. So from equation (61), it is obviously shown that the orthotropic rectangular plate with elastic end conditions and on constant elastic foundation and traversed by moving distributed force with constant speed reaches a state of resonance whenever

$\phi_s = \Omega_{rr}$ (98)

While equation (71) illustrates that the same orthotropic rectangular plate with elastic end conditions and on constant elastic foundation and traversed by moving distributed force with constant speed reaches a state of resonance when

$\phi_s = \epsilon_r$ (99)

where

$$\epsilon_r = \Omega_{rr} - \frac{1}{2\Omega_{rr}}\Big[\Omega_{rr}^2 \alpha \vartheta^* \Big(\varepsilon_6 + \frac{1}{4\pi}\Big(\sum_{f=1}^{\infty} \tau_7^* \frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty} \tau_8^* \frac{\sin(2f+1)\pi\rho}{2f+1}\Big)\Big) - s_i^2 \alpha \vartheta^* \Big(\varepsilon_8 + \frac{1}{4\pi}\Big(\sum_{f=1}^{\infty} \tau_{23}^* \frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty} \tau_{24}^* \frac{\sin(2f+1)\pi\rho}{2f+1}\Big)\Big)\Big] \quad (100)$$

Comparing equations (84) and (85), one obtains

$$\epsilon_r = \Omega_{rr}\Big[1 - \frac{1}{2\Omega_{rr}^2}\Big(\Omega_{rr}^2 \alpha \vartheta^* \Big(\varepsilon_6 + \frac{1}{4\pi}\Big(\sum_{f=1}^{\infty} \tau_7^* \frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty} \tau_8^* \frac{\sin(2f+1)\pi\rho}{2f+1}\Big)\Big) - s_i^2 \alpha \vartheta^* \varepsilon_8 + \frac{1}{4\pi}\Big(\sum_{f=1}^{\infty} \tau_{23}^* \frac{\cos(2f+1)\pi\rho}{2f+1} - \sum_{f=1}^{\infty} \tau_{24}^* \frac{\sin(2f+1)\pi\rho}{2f+1}\Big)\Big)\Big] = \Omega_{nn} \quad (101)$$

6 Graphs of the Numerical Solutions

To expatiate the analysis presented in this work, orthotropic rectangular plate is assumed to be of length $L_y =$

$0.923m$, breadth $L_x = 0.432m$ the load velocity c = 0.8123m/s and ρ = $0.4m$.The results are presented on the various plotted curves below for both clamped end conditions (classical end condition) and clamped elastic end conditions (non-classical end condition).

Figures 6.1 and 6.2 display the effect of rotatory inertia correction factor$\boldsymbol{R_o}$on the deflection profile of orthotropic rectangular plate elastically supported at all ends under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of rotatory inertia correction factor$\boldsymbol{R_o}$ increases.

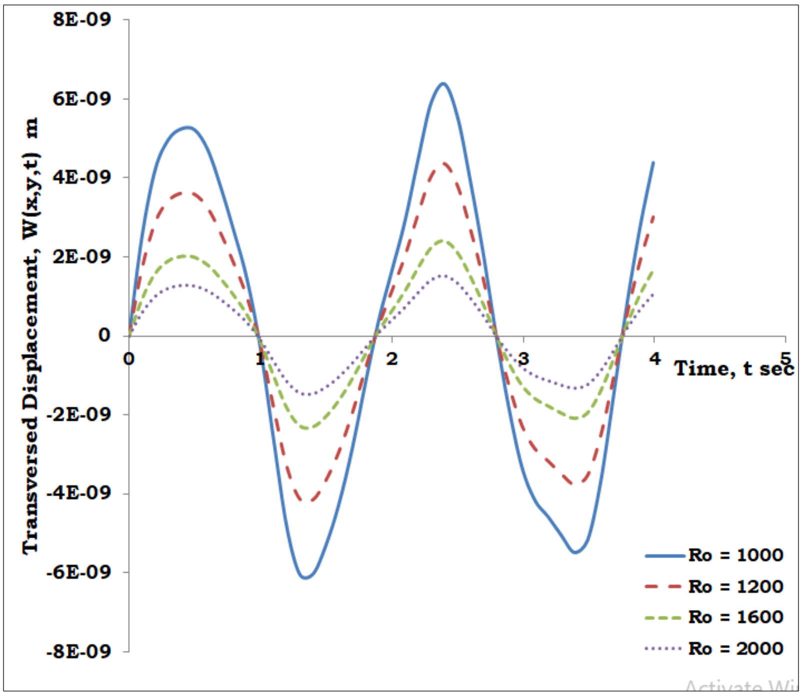

Figure 6.1: shows displacement Profile of Orthotropic Rectangular Plate with Varying R_o With elastically supported at all ends and Traversed by Moving Force

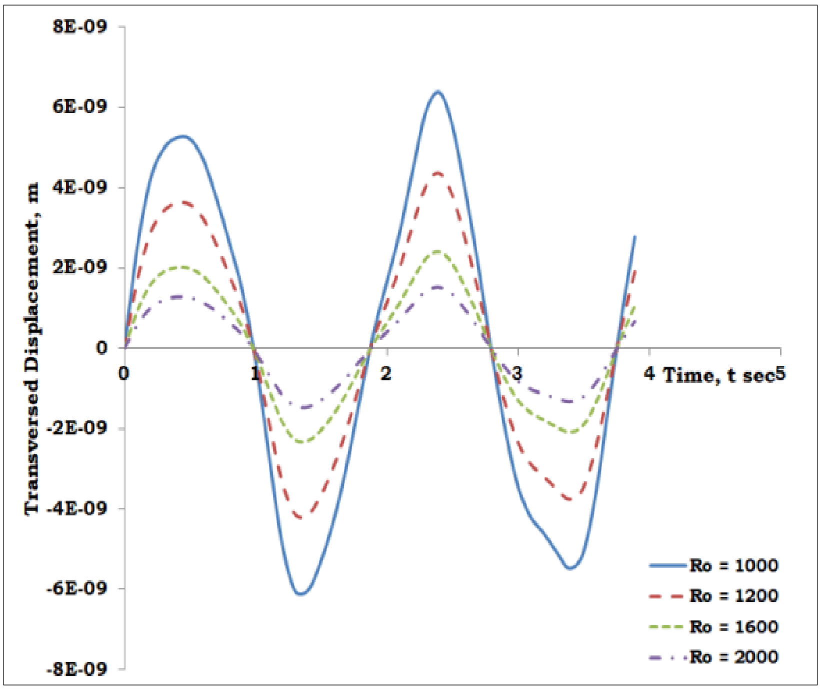

Figure 6.2: shows displacement Profile of Orthotropic Rectangular Plate with Varying R_o With elastically supported at all ends and Traversed by Moving Mass

Figures 6.3 and 6.4 display the effect foundation modulus$\boldsymbol{K_o}$ on the deflection profile of orthotropic rectangular plate elastically supported at all ends under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity $\boldsymbol{K_o}$ increases.

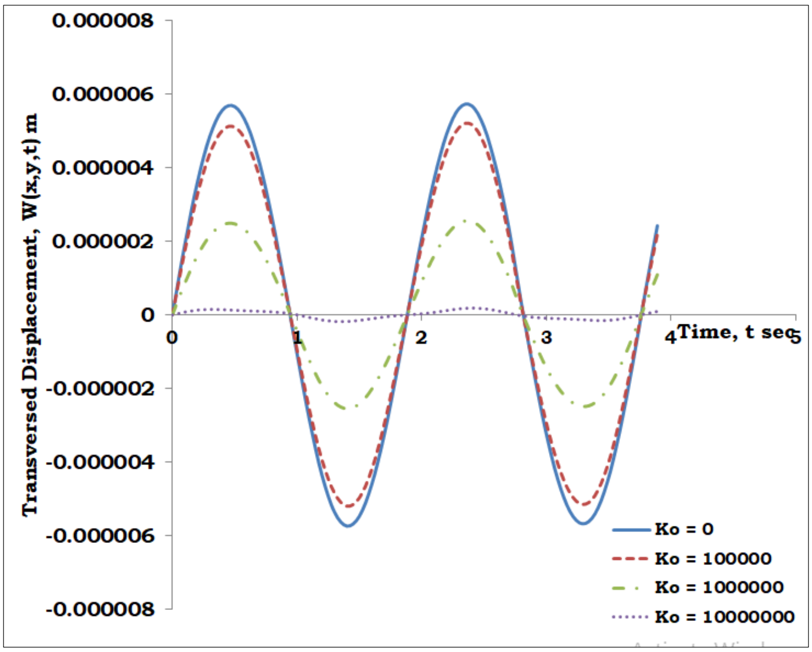

Figure 6.3: shows displacement Profile of Orthotropic Rectangular Plate with Varying K_o With elastically supported at all ends and Traversed by Moving Force

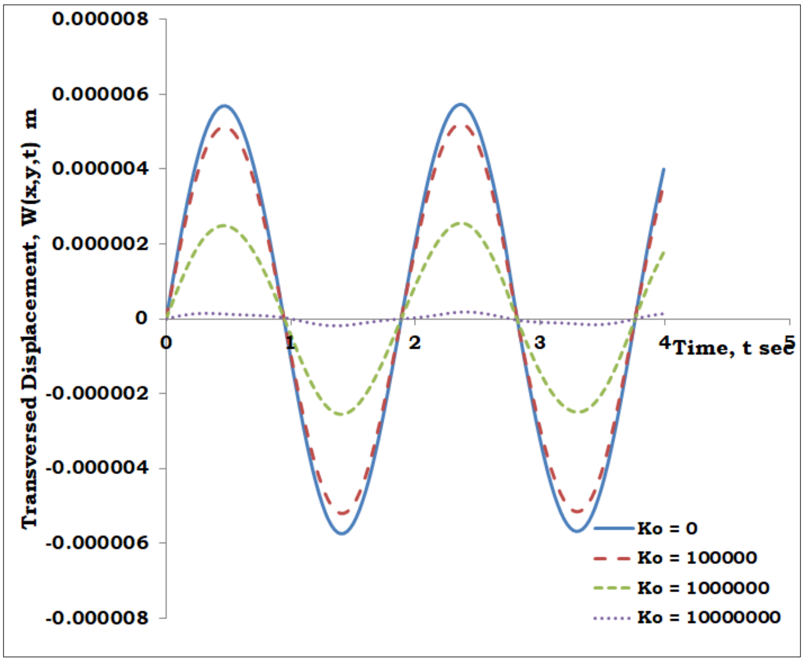

Figure 6.4: shows displacement Profile of Orthotropic Rectangular Plate with Varying K_o With elastically supported at all ends and Traversed by Moving Mass

Figures 6.5 and 6.6 display the effect of shear modulus $\boldsymbol{G_o}$ on the deflection profile of orthotropic rectangular plate elastically supported at all ends under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity $\boldsymbol{G_o}$ increases.

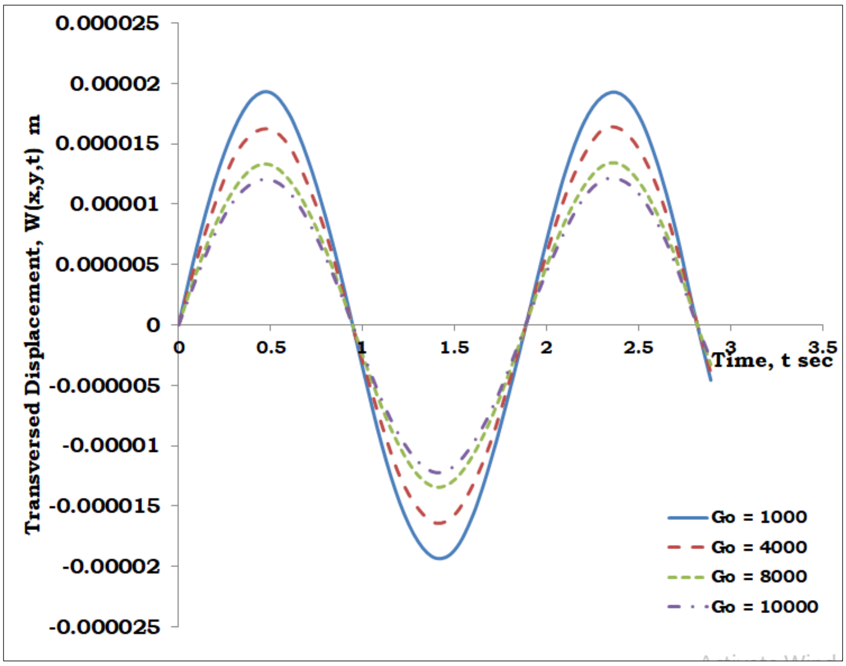

Figure 6.5: shows displacement Profile of Orthotropic Rectangular Plate with Varying G_o With elastically supported at all ends and Traversed by Moving Force

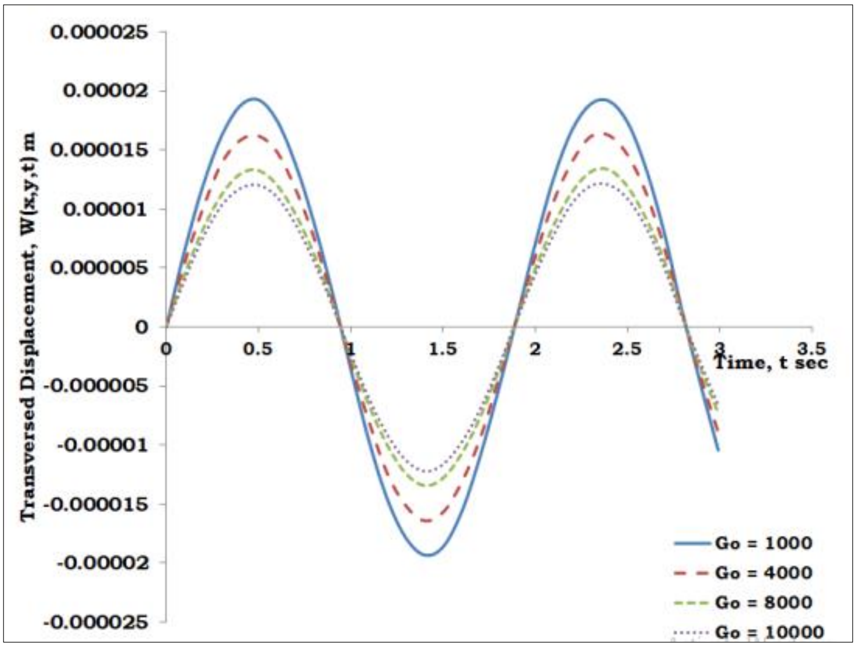

Figure 6.6: shows displacement Profile of Orthotropic Rectangular Plate with Varying G_o With elastically supported at all ends and Traversed by Moving Mass

7 CONCLUSION

Amplitude variation of moving distributed masses of orthotropic rectangular plate with elastically supported ends under moving loads has been examined in this research article. The closed form solutions to partial differential model with variable and singular coefficients of the orthotropic rectangular plates has been obtained for both cases of moving force and moving mass using the technique adopted by Shadnam, *et al.*, [19], was applied to remove the singularity in the governing fourth order partial differential equation and thereby transforming it to a sequence of second order ordinary differential equations. Using the asymptotic technique of Struble and Laplace transformation, the analytical solution is obtained. The solutions are then analysed. From the analysis, it was evidence that for the equal natural frequency, the critical speed for moving mass problem is smaller than that of moving force problem. Hence resonance is attained earlier in moving mass system than in the moving force system. The results in the plotted curves show that increase in rotatory inertia correction factor, R_o, foundation modulus K_o and shear modulus G_o resulted to decrease in the amplitudes of the orthotropic rectangular plates for both cases of moving force and moving mass problems. It is also depicted in the curves that the response amplitude of moving mass problem is higher than of moving force problem which indicates that resonance is reached earlier in moving mass problem than in moving force problem of the amplitude variation of moving distributed masses of orthotropic rectangular plate with elastically supported ends under moving loads.

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