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Original Research Article

Non-stationary Analysis of Elastically Supported Rayleigh Beam under the Circulation of Moving Distributed Masses on a Constant Subgrade to Arbitrary Varying Time

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Abstract

This study investigates the dynamic behaviour of an elastically supported uniform Rayleigh beam subjected to the passage of moving distributed masses with varying velocities, where the loading conditions vary arbitrarily with time. The motion of this problem is described by a fourth-order partial differential equation, which governs its behaviour. The beam's non-stationary response under such dynamic loading scenarios is analysed using the weighted residual method, which converts the governing equation into a sequence of linked second-order differential equations to facilitate the analysis. A rewritten version of Struble's asymptotic method further simplifies the transformed governing equation. This modification aids reduction in the complexity of the equation. The closed-form response is contrasted for the acceleration and deceleration motion. The study thoroughly examines how different velocities and frequencies of the moving force affect the dynamic behaviour of the beam. Key aspects explored include the influence of axial force, foundation modulus, and shear modulus in the support structure, the impact of varying mass distributions, and the time-dependent nature of the applied loads. The results help further understand the structural dynamics in complex environments and offer insights into optimising the design and performance of similar systems under non-stationary dynamic loads.

Keywords: Bi-parametric Elastic Foundation Partially Distributed Masses Axially Prestressed Thick Beam Dynamic Deflections Elastically Supported Beam. [2020] 37N15; 74K10; 74H15; 74H50.

1. INTRODUCTION

The vibration caused by vehicles travelling on structures like roads, bridges, and rails has become a significant and challenging issue due to the increasing speed capabilities of vehicles. The vibration problem in structural members, particularly in bridges and cable railways, when they experience moving loads such as trains, cars, ships, and lorries, is significant in many engineering applications. When the subsystem's inertia is ignored, the problem becomes a vibration of the distributed system with a given external moving force, called the moving force problem. However, when the subsystem's inertia is considered, and an infinite coupling stiffness is assumed between it and the distributed system, it becomes the moving mass problem. The large deflections and vibrations of beam-like structures, especially bridges, caused by high-speed and heavy vehicles can significantly increase the internal stresses and impact the safety and serviceability of the bridges. It is crucial to have a comprehensive understanding of the dynamic properties of these structures in order to design and operate them efficiently.

The vibration of beams is a multi-parameter phenomenon that has been studied extensively by researchers in applied mathematics and engineering. Various authors in this field have investigated the topic, including [1-8]. They have studied the vibration of beams due to applied forces, both moving and static. The issue of moving loads was first considered approximately by Stokes [9, 10] for the case of a girder with negligible mass compared to the mass of a single moving load of constant magnitude. Later, [11-13] examined the situation of a moving load whose mass was insignificant compared to the girder's mass. For the case of a load mass small against the beam mass [11], initially examined the problem for a supported beam and a constant concentrated force using the method of expansion of the eigenfunctions. [12-14] also solved it using Green's functions and integral equations. [15] tackled the problem of load and beam mass, which is considerably more complicated than the preceding exceptional cases. In his excellent monograph on this subject [16], provided an extended review of the topic. [3] studied the effects of the constant speed and damping on the response of a beam based on Fr`y ba's text. Moreover [17], used harmonic analysis to conduct a thorough investigation of the dynamic response of many kinds of railway bridges that steam locomotives cross.

Several researchers have found that some parameters, which are typically overlooked, significantly impact the dynamic behaviour of bridges. For instance, the type of velocity is a crucial factor when studying the vibrations caused by moving loads. Although there are only a few publications on this topic, the work of [18] is noteworthy, when he examined a bridge subjected to a concentrated load, the resulting dynamic response is affected by random velocities.

It is often challenging to solve non-classical boundary value problems using classical methods for dynamical problems. This is especially true when the problem involves moving loads, with or without considering the inertia effects of the moving load. [19] developed a method of separation of variables to solve Bernoulli-Euler beam vibration problems with time-dependent boundary conditions. [20] studied the deflection of beams on a two-parameter elastic foundation, while [21] investigated the behaviour of moving concentrated masses of simply rectangular plates on a variable Winkler foundation. Although their works were impressive, they only considered non-uniform beams subjected to a concentrated moving mass under the Winkler foundation, thus limiting the scope of the study. This is because the Winkler foundation predicts discontinuities in the deflections of the surface of the foundation at the end of the finite beam. In reality, the surface displacement continues beyond the load region. Some researchers have considered beams resting on variable elastic foundations but have yet to use the elastically supported boundary conditions used in this work.

The works mentioned earlier are based on structures with classical boundary conditions. In these structures, non-classical boundary conditions are not considered, loads are considered to be concentrated, and the foundation is assumed to be constant. However, this work focuses on the non-stationary analysis of an elastically supported Rayleigh beam under the circulation of moving distributed masses on a constant subgrade with varying velocities.

The remaining paper is organized in the following manner. Section 2 presents the mathematical formulation of the problem. Then, Section 3 discusses the discretization procedure. Next, Section 4 provides illustrative examples and a detailed discussion of the results. Finally, a concluding remark is presented in Section 5.

2. Governing Equation

The transverse vibration of a uniform elastic Rayleigh beam is described by the partial differential equation:

$$\frac{\partial^2}{\partial \eta^2}\left[\mathcal{E}\mathcal{J}\frac{\partial^2 \Theta_z(\eta,t)}{\partial \eta^2}\right] - \mathcal{N}_0\frac{\partial^2 \Theta_z(\eta,t)}{\partial \eta^2} + \mu_0\frac{\partial^2 \Theta_z(\eta,t)}{\partial t^2}$$
$$-\mu_0\mathcal{R}^0\frac{\partial^2 \Theta_z(\eta,t)}{\partial \eta^2 \partial t^2} + \mathcal{K}_0\Theta_z(\eta,t) - \mathcal{G}_0\frac{\partial^2 \Theta_z(\eta,t)}{\partial \eta^2} = f_z(\eta,t) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Equation (1) describes the deflection of a beam at a specific point in space, denoted by η, and time, denoted by t. The beam has various properties, including a constant moment of inertia of the beam cross-section, denoted by $\mathcal{J}$, a constant mass per unit length of the beam, denoted by μ_0, and a constant length, denoted by $\mathcal{L}$. The equation takes into account different parameters such as the modulus of elasticity, denoted by $\mathcal{E}$, the prestressed axial force, denoted by $\mathcal{N}_0$, the foundation modulus, denoted by $\mathcal{K}_0$, the shear rigidity, denoted by $\mathcal{G}_0$, the rotatory inertial, denoted by $\mathcal{R}^0$, and the deflection of the beam from its unloaded equilibrium point, denoted by the symbol $\Theta_z(\eta,t)$. Furthermore, the equation considers the travelling load of the beam, denoted by $Q_z(\eta,t)$. It is essential to note that this structure has zero bending moment and deflection at both ends.

$$\Theta_z(0,t) = 0\,;\ \Theta_z(L,t) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$\frac{\partial^2 \Theta_z(\eta,t)}{\partial t^2}\Big|_{\eta=0} = 0\,;\ \frac{\partial^2 \Theta_z(\eta,t)}{\partial t^2}\Big|_{\eta=L} = 0$$

and the initial conditions are given as:

$$\Theta_z(\eta,t)|_{t=0} = 0\,;\ \frac{\partial \Theta_z(\eta,t)}{\partial t}\Big|_{t=0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and the load

$$f_z(\eta,t) = \frac{\bar{m}g}{\chi}[H(\eta-(\tau(t))+\chi)$$
$$-H(\eta-(\tau(t))-\chi)][1-\frac{d^2}{dt^2}[\frac{\Theta_z(\eta,t)}{g}]] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

the property of the Heaviside unity function is expressed as:

$$H(\eta) = \begin{cases} 0 & \text{if } \eta < 0, \\ 1 & \text{if } \eta > 0, \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

regarding limiting case, $\chi \to 0$,

$$\delta(\eta-(\tau(t))) = \frac{1}{\chi}[H(\eta-(\tau(t))+\chi) - H(\eta-(\tau(t))-\chi)] \ldots\ldots\ldots\ldots (6)$$

assuming a quadratic form for the motion of the force along the priismatic thick beam yields $\tau(t) = \chi + \nu_0 t + \frac{1}{2}at^2$, where $\delta(\bullet)$ represents the Dirac delta function. At time $t = 0$, the force $Q_z = \bar{m}g$ acts at the point χ, with initial velocity ν_0 and constant acceleration a. Additionally, operator in equation (4) $\frac{d^2}{dt^2}$ is defined in (7)

$$\frac{d^2}{dt^2}[\cdot] = [\frac{\partial^2}{\partial t^2} + 2\frac{d}{dt}(\tau(t))\frac{\partial^2}{\partial\eta\partial t} + (\frac{d}{dt}(\tau(t)))^2\frac{\partial^2}{\partial\eta^2}$$
$$+\frac{d^2}{dt^2}(\tau(t))\frac{\partial}{\partial\eta}][\cdot] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Equation (1) can be simplified by taking into account equations (4) through (6) and (7)

$$\frac{\partial^2}{\partial\eta^2}[\mathcal{E}\mathcal{J}\frac{\partial^2\Theta_z(\eta,t)}{\partial\eta^2}] - \mathcal{N}_0\frac{\partial^2\Theta_z(\eta,t)}{\partial\eta^2} + \mu_0\frac{\partial^2\Theta_z(\eta,t)}{\partial\eta^2}$$
$$-\mu_0\mathcal{R}^0\frac{\partial^2\Theta_z(\eta,t)}{\partial\eta^2\partial t^2} + \mathcal{K}_0\Theta_z(\eta,t) - \mathcal{G}_0\frac{\partial^2\Theta_z(\eta,t)}{\partial\eta^2}$$
$$= \frac{\bar{m}g}{\chi}[H(\eta-(\tau(t))+\chi) - H(\eta-(\tau(t))-\chi)]$$
$$-\frac{\bar{M}}{\chi}[H(\eta-(\tau(t))+\chi) - H(\eta-(\tau(t))-\chi)]$$
$$[\frac{\partial^2\Theta_z(\eta,t)}{\partial t^2} + 2\frac{d}{dt}(\tau(t))\frac{\partial^2\Theta_z(\eta,t)}{\partial\eta\partial t} + (\frac{d}{dt}(\tau(t)))^2\frac{\partial^2\Theta_z(\eta,t)}{\partial\eta^2}$$
$$+\frac{d^2}{dt^2}(\tau(t))\frac{\partial\Theta_z(\eta,t)}{\partial\eta}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

?? governs the motion of a prismatic Rayleigh beam under axial compression and accelerating loads.

3. Discretization Procedure

Consider

$$\Theta_z(\eta,t) = \sum_{z=1}^{\infty} Y_z(t)U_z(\eta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

To solve problems involving mechanical vibrations, the Weighted Residual Method, a versatile technique, will be adopted.

$$U_z(\eta) = \sin\frac{\iota_z\eta}{L} + \Upsilon_{1m}\cos\frac{\iota_z\eta}{L} + \Upsilon_{2m}\sinh\frac{\iota_z\eta}{L} + \Upsilon_{3m}\cosh\frac{\iota_z\eta}{L} \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

The mode frequency ι_z and the constants Υ_{1m}, Υ_{2m}, and Υ_{3m} determine the $z-th$ normal mode of vibration of a uniform beam function. These constants can be obtained by substituting 9 with appropriate boundary conditions. It should be noted that for a simply supported beam, $\Upsilon_{1m} = \Upsilon_{2m} = \Upsilon_{3m} = 0$ and $\iota_z = z\pi$.

The technique for solving the problem requires that equation (8) is expressed as:

$$\Omega(\eta,t) = \sum_{z=1}^{\infty} Y_z(t)U_z(\eta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Equation (8) in view of equation (9) becomes:

$$\ddot{Y}_z(t) + \gamma_{mf}^2 Y_z(t) + \Gamma \sum_{m=1}^{\infty} J_6^{\star}[[U_z(\tau(t))U_\epsilon(\tau(t))$$
$$+\frac{\chi^2}{6}(U_z(\tau(t))U_\epsilon''(\tau(t)) + U_z''(\tau(t))U_\epsilon(\tau(t))$$
$$+((\tau(t)))'(\tau(t))U_\epsilon'(\tau(t)))]\ddot{Y}_z(t) + 2(v_0 + at)[U_z(\tau(t))U_\epsilon(\tau(t))$$
$$+\frac{\chi^2}{6}(U_z(\tau(t))U_\epsilon''(\tau(t)) + U_z''(\tau(t))U_\epsilon(\tau(t))$$
$$+((\tau(t)))'(\tau(t))U_\epsilon'(\tau(t)))]\dot{Y}_z(t) + (v_0 + at)^2[U_z''(\tau(t))U_\epsilon(\tau(t))$$
$$+\frac{\chi^2}{6}(U_z^{iv}(\tau(t))U_\epsilon(\tau(t)) + U_z''(\tau(t))U_\epsilon''(\tau(t))$$
$$+((\tau(t)))'''(\tau(t))U_\epsilon'(\tau(t)))]Y_z(t) + a[U_z'(\tau(t))U_\epsilon(\tau(t))$$
$$+\frac{\chi^2}{6}(U_z'''(\tau(t))U_\epsilon(\tau(t)) + U_z'(\tau(t))U_\epsilon''(\tau(t))$$
$$+((\tau(t)))''(\tau(t))U_\epsilon'(\tau(t)))]Y_z(t)]$$
$$= \frac{\overline{m}g}{\mu_0(J_1 - R_0 J_2)}[U_z(\tau(t)) + \frac{\chi^2}{6}U_z''(\tau(t))] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Where,

$$J_1 = \int_0^L U_z(\eta)U_\epsilon(x)dx \; J_2 = \int_0^L U_z''(\eta)U_\epsilon(x)$$
$$J_3 = \int_0^L U_z^{iv}(\eta)U_\epsilon(x)dx \; J_4 = \int_0^L U_z''(\eta)U_\epsilon(x)dx$$
$$J_5 = \int_0^L U_z(\eta)U_\epsilon(x)dx \; J_6 = \int_0^L U_z''(\eta)U_\epsilon(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$\Gamma = \frac{\overline{m}}{\mu_0 L} \text{ and } \gamma_{mf}^2 = \frac{\mathcal{E}\mathcal{J}J_3 - \mathcal{N}_0 J_4 + \mathcal{K}_0 J_5 - \mathcal{G}_0 J_6}{\mu_0(J_1 - \mathcal{R}^0 J_2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

the transformed equation governing the vibration problem of a prismatic beam is represented by equation (12). This beam is traversed by partially distributed accelerating masses. The non-homogeneous second-order ordinary differential equation holds for all variants of classical boundary conditions. This article will explore two exceptional cases of equation (12).

3.1 Beam under Moving Force

If the load's inertia effect is ignored, the equation of motion for the elastically supported Rayleigh beam under concentrated moving force may be derived. i.e $\Gamma^{\star} = 0$ in equation (12). In light of these modifications, the uncoupled equation (12) becomes

$$\ddot{Y}_z(t) + \vartheta_{mf}^2 Y_z(t) = \kappa_{mf}[\sin\frac{z\pi}{L}(\chi + v_0 t + \frac{1}{2}at^2)$$
$$+\frac{\chi^2}{6}\{-(\frac{z\pi(v_0 + at)}{L})^2 \sin\frac{z\pi}{L}(\chi + v_0 t + \frac{1}{2}at^2)$$
$$+\frac{z\pi a}{L}\cos\frac{z\pi}{L}(\chi + v_0 t + \frac{1}{2}at^2)\}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

solving equation (15), taking into account equations (3) and (11), one obtains:

$$\Theta(\eta, t) = \sum_{z=1}^{\infty} \frac{\kappa_{mf}}{\vartheta_{mf}}\{\frac{c_1 + c_2 i}{8}\sqrt{\frac{\pi}{aw}}[(i+1)$$
$$erf(\frac{(1-i)(wat + v_0 w - \vartheta_{mf})}{2\sqrt{aw}}) - erf(\frac{(1-i)(v_0 w - \vartheta_{mf})}{2\sqrt{aw}})$$
$$(\cos m_1 - i\sin m_1) + (i-1)(erf(\frac{(1+i)(wat + v_0 w - \vartheta_{mf})}{2\sqrt{aw}})$$
$$-erf(\frac{(1+i)(v_0 w - \vartheta_{mf})}{2\sqrt{aw}}))(\cos m_2 - i\sin m_2)]$$
$$-\frac{(c_1 - c_2 i)(1-i)}{8}\sqrt{\frac{\pi}{aw}}[erf(\frac{(1+i)(wat + v_0 w - \vartheta_{mf})}{2\sqrt{aw}})$$
$$-erf(\frac{(1+i)(v_0 w - \vartheta_{mf})}{2\sqrt{aw}})(\cos m_3 - i\sin m_3)$$

$$-(erf(\frac{(1+i)(wat+v_0w+\vartheta_{mf})}{2\sqrt{aw}})$$
$$-erf(\frac{(1+i)(v_0w+\vartheta_{mf})}{2\sqrt{aw}}))(\cos m_4+i\sin m_4)]]\times[\sin\frac{z\pi\eta}{L}] \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Equation (16) describes how a uniform Rayleigh beam on an elastic foundation responds to transverse displacement when it is subjected to partially distributed forces of varying velocities. The dynamic deflection of prestressed prismatic beam traversed by partially distributed loads with uniform velocity is obtained as:

$$\Theta(\eta,t)=\sum_{z=1}^{\infty}\frac{P_m}{\gamma_{mf}(\gamma_{mf}^4-\omega^4)}\{z_1(\gamma_{mf}^2+\omega^2)(\gamma_{mf}\sin\omega t-\omega\sin\gamma_{mf}t)$$
$$+z_3(\gamma_{mf}^2-\omega^2)(\gamma_{mf}\sinh\omega t-\omega\sin\gamma_{mf}t)$$
$$+z_2\gamma_{mf}(\gamma_{mf}^2+\omega^2)(\cos\omega t-\cos\gamma_{mf}t)$$
$$+z_4\gamma_{mf}(\gamma_{mf}^2-\omega^2)(\cosh\omega t-\cos\gamma_{mf}t)\}\times[\sin\frac{z\pi\eta}{L}] \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Where,

$$z_1=1-\frac{1}{6}(\frac{\chi\iota_z v_0}{L})^2, z_2=\Upsilon_{1m}-\frac{\Upsilon_{1m}}{6}(\frac{\chi\iota_z v_0}{L})^2$$
$$z_3=\Upsilon_{2m}+\frac{\Upsilon_{2m}}{6}(\frac{\chi\iota_z v_0}{L})^2, z_4=\Upsilon_{3m}+\frac{\Upsilon_{3m}}{6}(\frac{\chi\iota_z v_0}{L})^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

3.2 Beam under Moving Mass

The impact of the moving mass on the system is examined in this section. We emphasize on the instance in which the moving mass problem corresponds to $\Gamma^\star\neq 0$. Equation (12) has to be solved. In order to do this, we rearrange the equation (12) as follows:

$$\ddot{Y}_z(t)+\vartheta_{mf}^2Y_z(t)+\Gamma^\star\sum_{z=1}^{\infty}J_6^\star[[U_z(\tau(t))U_\epsilon(\tau(t))$$
$$+\frac{\chi^2}{6}(U_z(\tau(t))U_\epsilon''(\tau(t))+U_z''(\tau(t))U_\epsilon(\tau(t))$$
$$+((\tau(t)))'(\tau(t))U_\epsilon'(\tau(t)))]\ddot{Y}_z(t)+2(v_0+at)$$
$$[U_z(\tau(t))U_\epsilon(\tau(t))+\frac{\chi^2}{6}(U_z(\tau(t))U_\epsilon''(\tau(t))+U_z''(\tau(t))U_\epsilon(\tau(t))$$
$$+((\tau(t)))'(\tau(t))U_\epsilon'(\tau(t)))]\dot{Y}_z(t)+(v_0+at)^2[U_z''(\tau(t))U_\epsilon(\tau(t))$$
$$+\frac{\chi^2}{6}(U_z^{iv}(\tau(t))U_\epsilon(\tau(t))+U_z''(\tau(t))U_\epsilon''(\tau(t))$$
$$+((\tau(t)))'''(\tau(t))U_\epsilon'(\tau(t)))]Y_z(t)+a[U_z'(\tau(t))U_\epsilon(\tau(t))$$
$$+\frac{\chi^2}{6}(U_z'''(\tau(t))U_\epsilon(\tau(t))+U_z'(\tau(t))U_\epsilon''(\tau(t))$$
$$+((\tau(t)))''(\tau(t))U_\epsilon'(\tau(t)))]Y_z(t)]$$
$$=\kappa_{mf}[\sin\frac{z\pi}{L}(\chi+v_0t+\frac{1}{2}at^2)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

In what follows, a modification of a versatile asymptotic technique known as the Struble asymptotic method is employed to simplify the RHS of equation (19). Thus, to this effect, equation (19) reduces to:

$$\ddot{Y}_z(t)+\vartheta_{mf}^2Y_z(t)=\kappa_{mf}[\sin\frac{z\pi}{L}(\chi+v_0t+\frac{1}{2}at^2)] \ldots\ldots\ldots\ldots (20)$$

Where,

$$\kappa_1=U_1+\frac{\chi^2}{3}[U_2+U_3], \kappa_2=U_2+\frac{\chi^2}{6}[U_7+U_8+U_9],$$
$$\kappa_3=U_4+\frac{\chi^2}{6}[U_5+3U_6] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

and

$$\vartheta_{mm}=\vartheta_{mf}\{1-\frac{\Gamma^\star J_{6ss}^\star}{2}\{\kappa_1-\frac{1}{\vartheta_{mf}^2}\{v_0^2\kappa_2+a\kappa_3\}\}\} \ldots\ldots\ldots\ldots (22)$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. In account of the initial conditions, equation (20) becomes

$$\Theta(\eta,t)=\sum_{z=1}^{\infty}\frac{\kappa_{mf}}{\vartheta_{mm}}\{\frac{c_1+c_2 i}{8}\sqrt{\frac{\pi}{a\omega}}[(i+1)$$
$$erf(\frac{(1-i)(\omega at+\nu_0\omega-\vartheta_{mm})}{2\sqrt{a\omega}})-erf(\frac{(1-i)(\nu_0\omega-\vartheta_{mm})}{2\sqrt{a\omega}})$$
$$(\cos m_1-i\sin m_1)+(i-1)(erf(\frac{(1+i)(\omega at+\nu_0\omega-\vartheta_{mm})}{2\sqrt{a\omega}})$$
$$-erf(\frac{(1+i)(\nu_0\omega-\vartheta_{mm})}{2\sqrt{a\omega}}))$$
$$(\cos m_2-i\sin m_2)]-\frac{(c_1-c_2 i)(1-i)}{8}\sqrt{\frac{\pi}{a\omega}}$$
$$[erf(\frac{(1+i)(\omega at+\nu_0\omega-\vartheta_{mm})}{2\sqrt{a\omega}})-erf(\frac{(1+i)(\nu_0\omega-\vartheta_{mm})}{2\sqrt{a\omega}})$$
$$(\cos m_3-i\sin m_3)-(erf(\frac{(1+i)(\omega at+\nu_0\omega+\vartheta_{mm})}{2\sqrt{a\omega}})$$
$$-erf(\frac{(1+i)(\nu_0\omega+\vartheta_{mm})}{2\sqrt{a\omega}}))(\cos m_4+i\sin m_4)]]\times[\sin\frac{z\pi\eta}{L}] \text{ …………….. (23)}$$

The transverse displacement response of a supported uniform Rayleigh beam subjected to a dispersed moving mass with various velocities while resting on an elastic foundation is represented by equation (23). Similar to this, the beam's deflection under loads moving at a constant speed may be obtained as:

$$\Theta(\eta,t)=\sum_{z=1}^{\infty}\frac{P_{mm}}{\vartheta_{mm}(\vartheta_{mm}^4-\omega^4)}\{z_1(\vartheta_{mm}^2+\omega^2)(\vartheta_{mm}\sin\omega t-\omega\sin\vartheta_{mm}t)$$
$$+z_3(\vartheta_{mm}^2-\omega^2)(\vartheta_{mm}\sinh\omega t-\omega\sin\vartheta_{mm}t)$$
$$+z_2\vartheta_{mm}(\vartheta_{mm}^2+\omega^2)(\cos\omega t-\cos\vartheta_{mm}t)$$
$$+z_4\vartheta_{mm}(\vartheta_{mm}^2-\omega^2)(\cosh\omega t-\cos\vartheta_{mm}t)\}\times[\sin\frac{z\pi\eta}{L}]$$

and

$$\vartheta_{mm}=\vartheta_{mf}\{1-\frac{\Gamma^\star J_{6ss}^\star}{2}\{\varsigma_1-\frac{1}{\vartheta_{mf}^2}\{\nu_0^2\varsigma_2+a\varsigma_3\}\}\} \text{ …………………… (24)}$$

Where,

$$\varsigma_1=U_1+\frac{\chi^2}{3}[U_2+U_3], \varsigma_2=U_2+\frac{\chi^2}{6}[U_7+U_8+U_9],$$
$$\varsigma_3=U_4+\frac{\chi^2}{6}[U_5+3U_6] \text{ …………………………………………… (25)}$$

Where,

$$P_{mm}=\frac{\Gamma^* gL}{J_7} \text{ ………………... (26)}$$

The resonance phenomenon is closely linked to a dynamic system like the one we are currently examining. Therefore, analysing the conditions that can lead to resonant and maximum amplitude is necessary. The velocity of the moving load, at which a resonance effect happens in the undamped system, is considered a critical speed. Equation (17) demonstrates that when a moving force crosses a prismatic Rayleigh beam with an elastically supported boundary condition, resonance is reached when

$$\gamma_{mf}=\omega \text{ ……………..….. (27)}$$

according to the equation $\omega=\frac{\iota_z\nu_0}{L}$, the critical speed ν_{cr} of the moving force system is given by $\nu_{cr}=\frac{\gamma_{mf}L}{\iota_z}$. When the system is under a moving mass, the corresponding resonance condition is shown in equation (24).

$$\vartheta_{mm}=\omega \text{ ……………….. (28)}$$

that is

$$\vartheta_{mf}\{1-\frac{\Gamma^\star J_{6ss}^\star}{2}\{\varsigma_1-\frac{1}{\vartheta_{mf}^2}\{\nu_0^2\varsigma_2+a\varsigma_3\}\}\}=\omega \text{ ……………….……… (29)}$$

which implies that
$\vartheta_{mf} = \frac{\iota_z v_0}{L}$ ………………………. (30)

Equation (27) shows that a beam subject to moving distributed masses experiences resonance effects whenever:
$\vartheta_{mm} = \frac{\iota_z v_0}{L}$ ………………..……. (31)

from equation (26)
$\frac{\iota_z v_0}{L} = \vartheta_{mf}\{1 - \frac{\Gamma^\star J^\star_{6ss}}{2}\{\varsigma_1 - \frac{1}{\vartheta^2_{mf}}\{v_0^2\varsigma_2 + a\varsigma_3\}\}\}$ …………… (32)

which implies
$\vartheta_{mf} = \frac{\frac{\iota_z v_0}{L}}{\{1 - \frac{\Gamma^\star J^\star_{6ss}}{2}\{\varsigma_1 - \frac{1}{\vartheta^2_{mf}}\{v_0^2\varsigma_2 + a\varsigma_3\}\}\}}$ ……………………….….. (33)

evidently, for the same natural frequency, the critical velocity for the system consisting of an elastically supported Rayleigh beam is greater than that of the moving mass. Resonance is reached sooner in the moving mass system than in the moving force for the same natural frequency of an elastic beam. This result perfectly agrees with existing results; see ref [22, 23]. In the same vein, the solutions of equation (16) and equation (23) show that as the axial force, foundation modulus, rotatory inertia correction factor and other structural parameters increase, the critical speeds of the thick beam increase, consequently reduce the speed of resonance. It has been deduced that the critical speed for an elastically supported beam carrying a moving mass is smaller than that of the same beam carrying a moving force, as indicated by equations (27) and (33). This deduction highlights that it is not safe to assume that the response of the moving force can be used as an approximation for the response to the moving mass.

4. Illustrative Examples, Results and Discussion

4.1 Illustrative Examples

This section presents practical examples of the analysis, taking into consideration non-classical boundary conditions, especially elastically supported boundary conditions.

4.1.1 Simple-elastic Boundary Conditions

As an example, we consider a prismatic Rayleigh beam simply supported at the end η and elastically supported at the end $\eta = L$, the conditions are expressed as follows:
$\Theta_z(0,t) = 0 = \Theta''_z(0,t)$…………………………………... (34)

at the end $\eta = 0$

and
$\Theta''_z(L,t) - k_1\Theta'_z(L,t) = 0 = \Theta'''_z(L,t) + k_2\Theta_z(L,t)$ ……. (35)

at the other end $\eta = L$ for the normal modes:
$U_z(0) = 0 = U''_z(0)$ ……………………………………… (36)

at $\eta = 0$ and
$U''_z(L) - k_1U'_z(L) = 0 = U'''_z(L) + k_2U_z(L)$ ……………. (37)

at $\eta = L$ which implies
$U_\epsilon(0) = 0 = U''_\epsilon$ ………………………………………….. (38)

and
$U''_\epsilon(L) - k_1U'_\epsilon(L) = 0 = U'''_\epsilon(L) + k_2U_z(L)$ …………..... (39)

thus it can be shown that $\Upsilon_{1m} = \Upsilon_{3m} = 0$ and
$\Upsilon_{2m} = \frac{k_1\cos\iota_z + \frac{\iota_z}{L}\sin\iota_z}{\frac{\iota_z}{L}\sinh\iota_z - k_1\cosh\iota_z} = \frac{\frac{\iota_z^3}{L^3}\cos\iota_z - k_2\sin\iota_z}{\frac{\iota_z^3}{L^3}\cosh\iota_z + k_2\sinh\iota_z}$ ……………..…… (40)

and from equation (40), one obtains
$\tan\iota_z = \tanh\iota_z$ ……………………………………..….… (41)

as the frequency equation for the dynamical problem, and we have
$\iota_1 = 3.927, \iota_2 = 7.069, \iota_3 = 10.210$ …………….. (42)

using equations (40), (41) and (42) in equations (16) and (23), one obtains the displacement response respectively to a moving force and moving mass of a simply-elastic ends prismatic Rayleigh beam with variable velocity and constant foundation.

4.2 RESULTS AND DISCUSSION

This illustration is done by considering a homogenous beam of modulus of elasticity $\mathcal{E} = 3.1 \times 10^{10} N/m^2$, the moment of inertial $\mathcal{J} = 2.87698 \times 10^-3m^4$, the beam span $\mathcal{L}. = 50m$ and the mass per unit length of the beam $\mu_0 = 2758.291kg/m$. The values of foundation moduli is varied between $0N/m^3$ and $40000N/m^3$, the values of axial force $\mathcal{N}_0$ is varied between $0N$ and $2.0 \times 10^8 N$.

4.2.1 Effect of varying $\mathcal{N}_0$

Figure 1 demonstrates the effect of varying $\mathcal{N}_0$ on the dynamic deflection of the thick beam. The dynamic deflection is observed to reduce significantly with increased axial force. In addition, for the moving force, which is Figures 1(a) and 1(b), maximum dynamic deflection is recorded at a much later time of the beam and is higher at the deceleration motion. A similar result is also obtained when the thick beam is traversed by moving mass. This we observed in figures 1(c) and 1(d). Maximum deflection appears to have taken place with the accelerated motion.

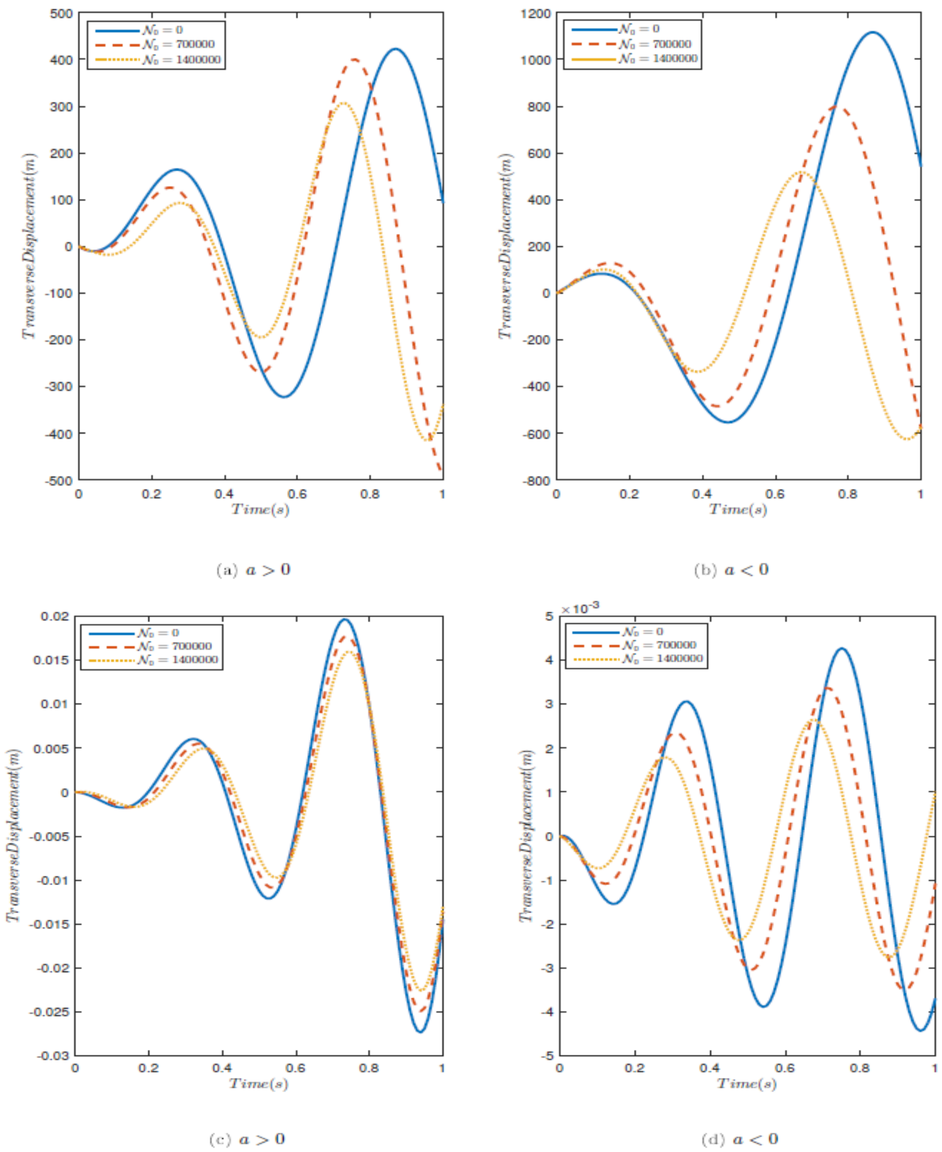


Figure 1: Effects of axial force $\mathcal{N}_0$ on the dynamic deflections of elastically Supported Rayleigh beam

4.2.2 Effect of varying $\mathcal{K}_0$

A look at figure 2 shows the effect of the beam foundation on the elastically supported beam. Figures 2(a) and 2(b) shows the Rayleigh beam when traversed by a moving force. It was shown that as the foundation stiffness $\mathcal{K}_0$ increases, the structural system becomes rigid and the response amplitude decreases. The same results is also obtained when traversed by moving mass as seen in Figures 2(c) and 2(d). A general trend in Figure 2 is that displacement responses decreases as the foundation modulus $\mathcal{K}_0$ spacing increases.

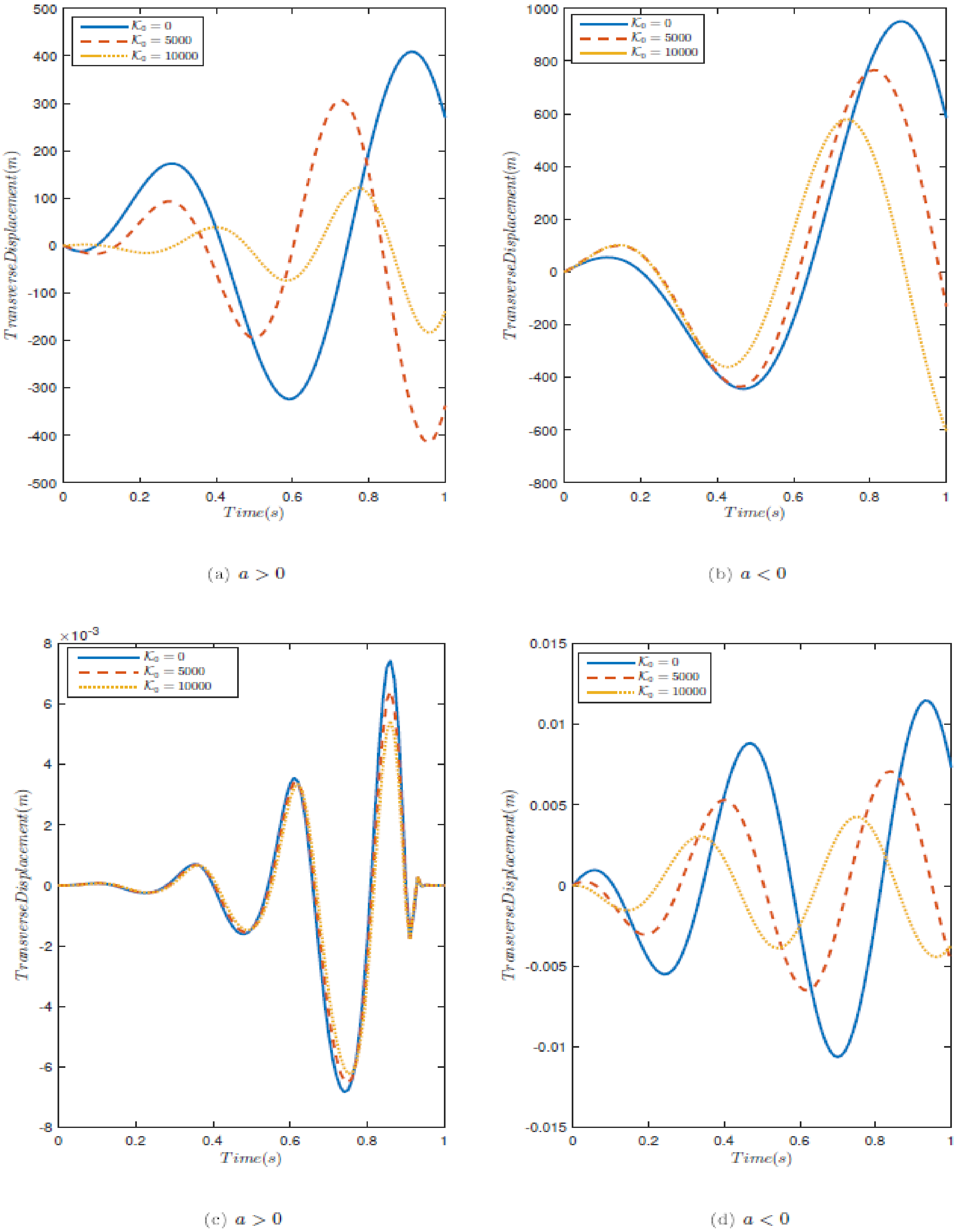


Figure 2: Effects of foundation modulus $\mathcal{K}_0$ on the dynamic deflections of elastically Supported Rayleigh beam

4.2.3 Effect of varying $\mathcal{G}_0$

In order to have an overview of the space-time evolution of the dynamic response of a thic beam to various shear modulus. How the shear modulus $\mathcal{G}_0$ of the Rayleigh beam, affects the dynamics deflection of the dynamical system is shown graphically in Figure 3. It is seen in Figures 3(a) and 3(b) that the dynamic deflection decreases when the shear modulus is increased. The same result is obtained when the same beam is traversed by a moving mass in Figures 3(c) and 3(d). It is observed that as $\mathcal{G}_0$ increases, the becomes more flexible and the dynamic deflection decreases.

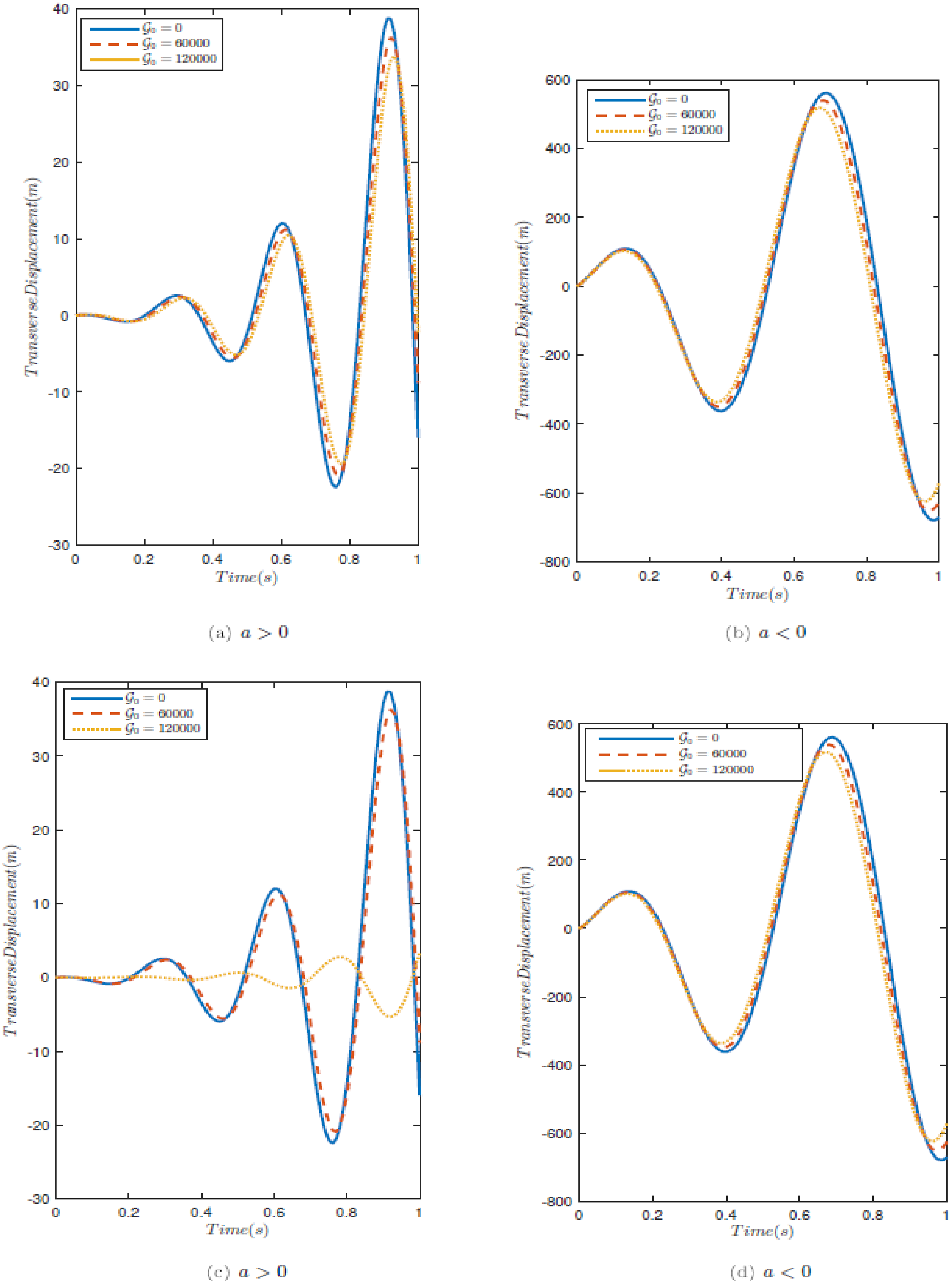
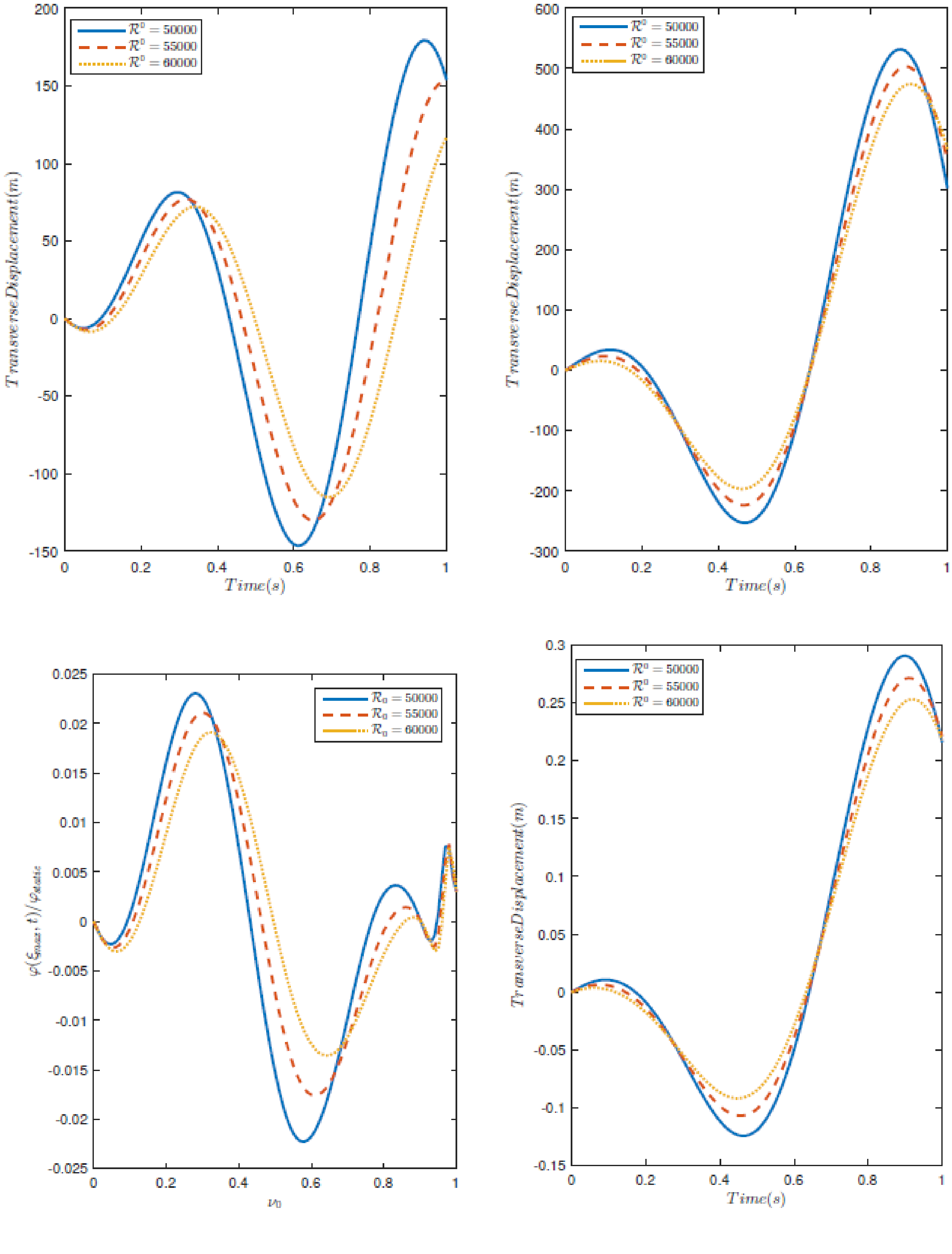


Figure 3: Effects of shear modulus $\mathcal{G}_0$ on the dynamic deflections of elastically Supported Rayleigh beam

4.2.4 Effect of varying $\mathcal{R}^0$

The effect of rotatory inertia correction factor, $\mathcal{R}^0$, on the dynamic deflection amplitude of the uniform Raleigh beam is illustrated in Figure 4. Figures 4(a) and 4(b) demonstrate the deflection curves the prismatic Rayleigh beam under the influence of accelerating forces and masses, respectively. It is shown that increasing the value of the rotatory inertia factor leads to a significant decreases the deflection amplitude. The same result is obtained when the prismatic thick beam is traversed by a moving mass in Figures 4(c) and 4(d).



(c) $a > 0$ (d) $a < 0$

Figure 4: Effects of rotatory inertia correction factor $\mathcal{R}_0$ on the dynamic deflections of elastically Supported Rayleigh beam

4.2.5 Effect of varying ν_0

In Figure 5, the effect of the different values of velocity ν_0 on the dynamical behaviour of the ellastically supported beam is demonstrated. Independent of the type of motion, a look at Figures 5(a) and 5(b) show that the maximum amplitudes of the deflections of the beam decreases with increasing value of ν_0. This response decreases with increasing the speed of the moving load, since the acting time on the load becomes shorter. Figure 5(c) has a higher maximum response than Figure 5(d). The maximum response is reached at an earlier time in figure 5(c) than the decelared motion.

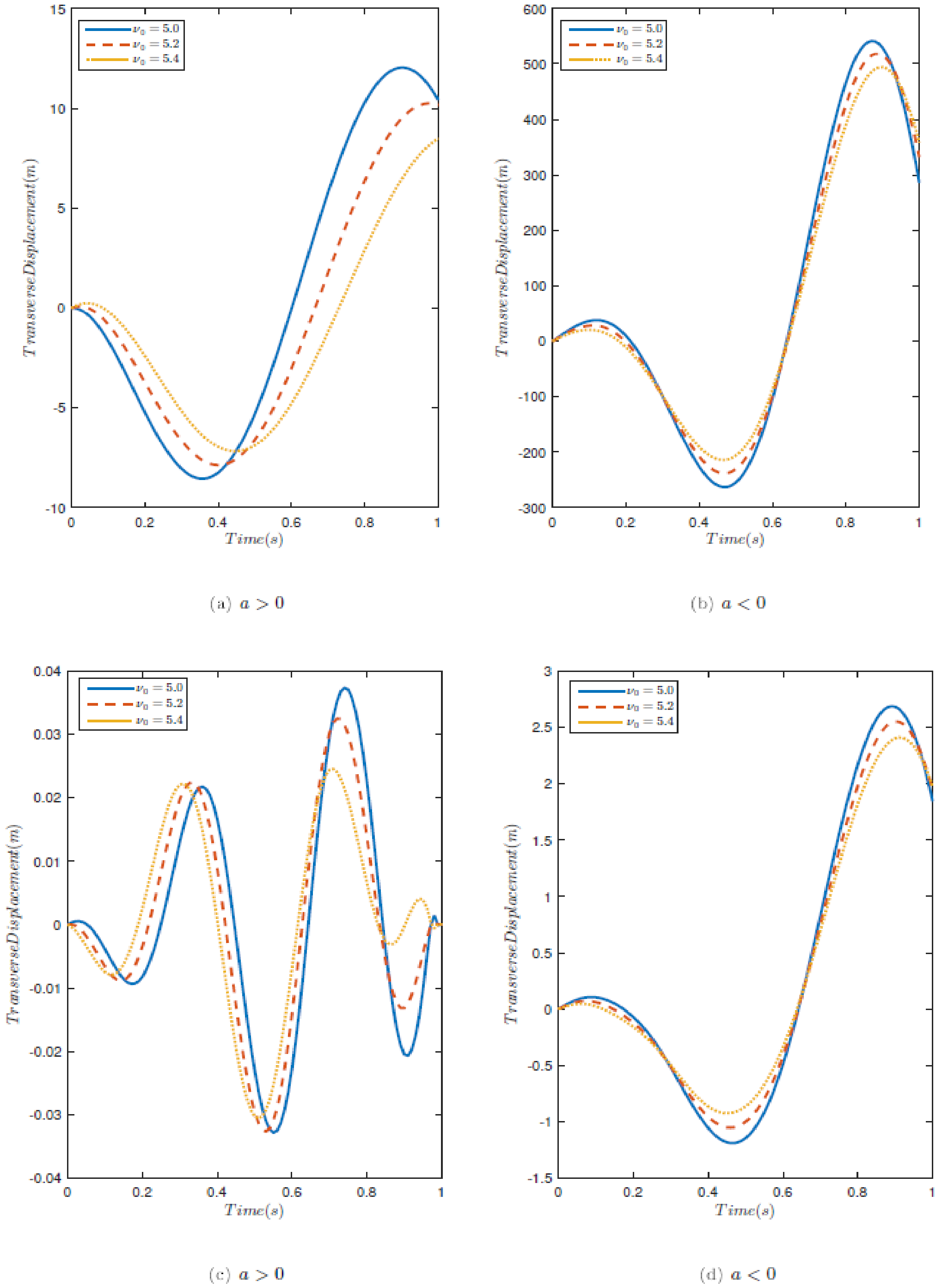


Figure 5: Effects of varrying velocity ν_0 on the dynamic response of elastically Supported Rayleigh beam

5. CONCLUSION

This research focuses on the non-stationary analysis of elastically supported homogenous isotropic prismatic Rayleigh beam under the circulation of moving distributed masses on a constant subgrade with varying velocities. The study shows that the load's inertial effect causes the set of differential equations of motion to be coupled. Ignoring this effect results in solving a set of uncoupled linear second-order differential equations, which is the solution for the corresponding moving distributed force and not the mass problem. The technique of Struble is employed to solve the governing differential equation, which yields a close-form solution of the governing fourth-order partial differential equation with variable and singular coefficients of uniform Rayleigh beam for the moving force problem. The solutions are analyzed, and resonance conditions are obtained for the problem. The plotted curves only depict the effects of axial force, shear modulus, and foundation modulus on the beam for the moving force problem.

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REFERENCES

1. Lee, H. (1996). Dynamic response of a beam with a moving mass.
2. Frýba, L. (1973). Simply supported beam subjected to a moving constant force. *Vibration of solids and structures under moving loads*, 13-32.
3. Frýba, L. (1976). Non-stationary response of a beam to a moving random force. *Journal of sound and vibration*, *46*(3), 323-338.
4. Adeoye, S. A., & Adeloye, T. O. (2024). On the dynamic characteristics of orthotropic rectangular plates under the infuence of moving distributed masses and resting on a variable elastic pasternak foundation. *Journal of Material Science and Manufacturing Research*, *5*(1), 1-19.
5. Esmailzadeh, E., & Ghorashi, M. (1995). Vibration analysis of beams traversed by uniform partially distributed moving masses. *Journal of sound and vibration*, *184*(1), 9-17.
6. Omolofe, B., Awodola, T. O., & Adeloye, T. O. (2018). Damping influence on the critical velocity and response characteristics of structurally prestressed beam subjected to travelling harmonic load. *Math Nat Sci Int J*, *3*, 18-28.
7. Adeloye, T. O., & Omolofe, B. (2024). Flexural Response of a Cantilever Beam under Masses of Varying Velocities with General Boundary Condition. *Journal of Mathematical Analysis and Modeling*, *5*(2), 14-43.
8. Adeloye, T. O. (2024). Dynamic Characteristics of a Simply Supported Beam Subjected to Compressive Axial Force and Impact Loads. *International Journal of Maritime and Interdisciplinary Research*, *4*(2), 203-220.
9. Stokes, S. G. G. (1849). Discussion of a differential equation relating to the breaking of railway bridges, Printed at the Pitt Press by John W. Parker.
10. Michaltsos, G. (2002). Dynamic behaviour of a single-span beam subjected to loads moving with variable speeds. *Journal of sound and vibration*, *258*(2), 359-372.
11. Krylov, A. (1905). Mathematical collection of papers of the academy of sciences, Mathematische Annalen, 61, 211.
12. Timoshenko, S. (1908). Forced vibration of prismatic bars, Izvestiya Kievskogo politekhnicheskogo instituta, 59, 163-203.
13. Lowan, A. N. (1935). Liv. on transverse oscillations of beams under the action of moving variable loads, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 19(127), 708-715.
14. Bondar, N. (1954). Dynamic calculations of beams subjected to a moving load, Issledovaniya po teorii sooruzhenii, 6, 11-23.
15. Bolotin, V. (1961). Problem of bridge vibration under the action of a moving load, Izvestiya an USSR, Mechanika, 1, 104-105.
16. Fr`yba, L. (1972). Vibration of solids and structures under moving load, research institute of transport.
17. Inglis, C. E. (2015). A mathematical treatise on vibrations in railway bridges, Cambridge University Press.
18. Zibdeh, H. (1995). Stochastic vibration of an elastic beam due to random moving loads and deterministic axial forces. *Engineering structures*, *17*(7), 530-535.
19. Mindlin, R., & Goodman, L. (1950). Beam vibrations with time-dependent boundary conditions.
20. Omolofe, B., & Oni, S. T. (2015). Transverse motions of rectangular plates resting on elastic foundation and under concentrated masses moving at varying velocities. *Latin American Journal of Solids and Structures*, 12, 1296-1318.
21. Oni, S., & Awodola, T. (2010). Dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-euler beam on variable elastic foundation. *Latin American Journal of Solids and Structures*, 3-20.
22. Gbadeyan, J., & Oni, S. (1992). Dynamic response to moving concentrated masses of elastic plates on a non- Winkler elastic foundation. *Journal of sound and vibration*, *154*(2), 343-358.
23. Gbadeyan, J., & Oni, S. (1995). Dynamic behaviour of beams and rectangular plates under moving loads. *Journal of sound and vibration*, *182*(5), 677-695.