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The recent development of non-monotone trust region methods

Baowei Liu
Department of Mathematics, Cangzhou Normal University, Cangzhou, Hebei Province, China

***Corresponding Author:**
Baowei Liu
Email: calbw@sina.cn

Abstract: Trust region methods are a class of numerical methods for optimization. They compute a trial step by solving a trust region sub-problem where a model function is minimized within a trust region. In this paper, we review recent results on non-monotone trust region methods for unconstrained optimization problems. Generally, non-monotone trust region algorithms with non-monotone technique are more effective than the traditional ones, especially when coping with some extreme nonlinear optimization problems. Results on trust region sub-problems and regularization methods are also discussed.

Keywords: non-monotone trust region method; unconstrained optimization; global convergence.

INTRODUCTION

Consider the following unconstrained optimization problem:

$$\min_{x\in R^n} f(x), \tag{1.1}$$

where $f(x): R^n \to R$ is a continuously differentiable function.

Trust region methods are a prominent class of iterative methods to solve the unconstrained optimization problem (1.1). A basic trust region algorithm works as follows. For a given iteration point x_k, it obtains a trial step d_k by solving the following quadratic sub-problem:

$$\begin{aligned} &\min q_k(d) = f_k + g_k^T d + \tfrac{1}{2} d^T B_k d, \\ &s.t. \|d\| \le \Delta_k, \end{aligned} \tag{1.2}$$

where $f_k = f(x_k)$, $g_k = \nabla f(x_k)$, $B_k \in R^{n\times n}$ is a symmetric matrix which is the Hessian matrix or its approximation of $f(x)$ at the current point x_k, $\Delta_k > 0$ is called the trust radius and $\|\cdot\|$ refers to the 2-norm. The ratio r_k between the actual reduction in the function value $f(x_k) - f(x_{k+1})$ and the predicted reduction $q_k(0) - q_k(d_k)$ plays a key role to decide whether the trial step is acceptable or not and how to adjust the trust region radius.

The usual trust region methods generate a sequence $\{x_k\}$ such that $\{f(x_k)\}$ is monotonically decreasing. However, a lot of numerical experiments indicate that enforcing monotonicity of $\{f(x_k)\}$ may considerably slow the rate of convergence when the iteration is trapped near a narrow valley [2, 3]. In order to overcome this drawback, the application of the non-monotone can be useful [1, 2, 3, 11, 15].

NON-MONOTONE TECHNIQUES

In 1986, Grippo et al. [4] proposed a non-monotone line search for Newton's method, in which the step-size α_k satisfies the following inequality:

$$f(x_k + \alpha_k d_k) \le f(x_{l(k)}) + \beta \alpha_k \nabla f(x_k)^T d, \tag{2.1}$$

where $\beta \in (0,1)$, $f(x_{l(k)}) = \max_{0\le j\le m_k} f(x_{k-j})$, $m_0 = 0$, $0 \le m_k \le \min\{m_{k-1}+1, M\} (k \ge 1)$, and $M \ge 0$ is an integer. The sequence $\{f(x_k)\}$ is non-increasing. Since then, the non-monotone technique has been exploited by many researchers [5-7] and a lot of numerical tests have showed that the non-monotone technique proposed by Grippo et al. [4] is efficient.

However, the non-monotone technique based on Grippo et al.'s approach has some disadvantages. For example, it follows from (2.1) that a good function value generated at any iteration may be thrown away due to the maximum. Zhang and Hager [8] have pointed out that in some cases, the numerical results are dependent on the choice of parameter M . In order to overcome these disadvantages, Zhang and Hager [8] proposed another non-monotone line search in 2004. They replaced the maximum function value in (2.1) with an average of function values and their non-monotone technique requires decreasing of an average of the successive function values. In detail, their method finds a step-size α_k satisfying the following condition:

$$f(x_k+\alpha_k d_k)\le C_k+\beta\alpha_k\nabla f(x_k)^T d, \tag{2.2}$$

where

$$C_k=\begin{cases} f(x_k), & k=0,\\ \frac{\eta_{k-1}Q_{k-1}C_{k-1}+f(x_k)}{Q_k}, & k\ge 1\end{cases},\qquad Q_k=\begin{cases}1, & k=0,\\ \eta_{k-1}Q_{k-1}+1, & k\ge 1,\end{cases} \tag{2.3}$$

and $\eta_{k-1}\in[\eta_{\min},\eta_{\max}]$, $\eta_{\min}\in[0,1)$ and $\eta_{\max}\in[\eta_{\min},1)$ are two chosen parameters. Numerical results showed that this non-monotone technique was superior to (2.1).

In 2008, Gu and Mo [9] introduced another non-monotone strategy. They replaced C_k in (2.2) with D_k

$$D_k=\begin{cases} f_k, & k=0,\\ \eta_k D_{k-1}+(1-\eta_k)f_k, & k\ge 1\end{cases} \tag{2.4}$$

for $\eta_k\in[\eta_{\min},\eta_{\max}]$.This non-monotone technique is efficient and robust which is showed by numerical experiments in [9].

Recently, in 2012, M. Ahookhosh et al. [10] introduced a more relaxed non-monotone strategy as following:

$$R_k=\eta_k f_{l(k)}+(1-\eta_k)f_k \tag{2.5}$$

in which $\eta_{k-1}\in[\eta_{\min},\eta_{\max}]$. This non-monotone technique is efficient and robust which is showed by numerical experiments in [10].

Non-monotone trust region methods

For the trust region method, the key problem is how to solve the trust region sub-problem (1.2). In 1993, Deng et al. [2] generalized the non-monotone proposed by Grippo et al. [4] to the trust region method, and proposed a non-monotone trust region method for unconstrained optimization. Theoretical analysis and numerical results show that algorithms with non-monotone technique are more effective than algorithms without it.

Non-monotone adaptive trust region methods

Adaptive trust region methods are very popular for solving the unconstrained optimization. Zhang et al. [18] proposed an adaptive trust region method, they construct a new trust region sub-problem, in which the trust region radius uses the information of g_k and B_k , i.e., they solve the following sub-problem:

$$\begin{aligned}&\min q_k(d)=f_k+g_k^T d+\tfrac{1}{2}d^T B_k d,\\ &s.t.\ \|d\|\le\Delta_k,\end{aligned} \tag{3.1}$$

where $\Delta_k=c^p\|g_k\|\bar{M}_k$, $0<c<1$, $\bar{M}_k=\|\hat{B}_k^{-1}\|$, and p is a nonnegative integer. $\hat{B}_k$ is a safely positive definite matrix based on Schnabel and Eskow [19] modified Cholesky factorization, $\hat{B}_k=B_k+E_k$, where $E_k=0$ if B_k is safely positive definite, and E_k is a diagonal matrix chosen to make $\hat{B}_k$ positive definite otherwise. This method defines Δ_k automatically based on g_k and B_k, while Δ_k in traditional trust region methods is independent of g_k and B_k .

In 2003, Zhang et al. [20] combined the adaptive trust region method with a non-monotone technique [4] to propose an adaptive non-monotone trust region method and obtained better numerical results than the former article [18] due to using the non-monotone. Two years later, Fu and Sun [3] combined Zhang's adaptive trust region method with a non-monotone technique [4], and constructed a new non-monotone adaptive trust region. Instead of solving the sub-problem (3.1) exactly, they solve the sub-problem (3.1) inexactly by using the truncated conjugate gradient method. However, the adaptive method needs the inverse of matrix B_k , hence it is appropriate only for small scale problems.

Motivated by Zhang's strategy, Shi and Guo [21] proposed a new adaptive radius for the trust region method. They choose μ, $\rho \in (0,1)$, and q_k to satisfy the following inequality

$$-\frac{g_k^T q_k}{q_k^T B_k q_k} \geq \tau \tag{3.2}$$

where $\tau \in (0,1]$, and set

$$d_k = -\frac{g_k^T q_k}{q_k^T \hat{B}_k q_k} \tag{3.3}$$

in which $\hat{B}_k$ is generated by the procedure: $q_k^T \hat{B}_k q_k = q_k^T B_k q_k + i\|q_k\|^2$, and i is the smallest nonnegative integer such that

$$q_k^T \hat{B}_k q_k = q_k^T B_k q_k + i\|q_k\|^2 > 0$$

So, they proposed a new trust region radius as follows

$$\Delta_k = \alpha_k \|g_k\| \tag{3.4}$$

where $\alpha = \rho^p d_k$, and p is the least positive integer number so that $r_k \geq \mu$.

Due to the fact that Shi's adaptive trust region method is more efficient than Zhang's adaptive trust region method. In 2010, M. Ahookhosh et al. [22] incorporate Shi's adaptive trust region method with a non-monotone technique [4] in order to propose the new non-monotone trust region method with an adaptive radius. They proved that the new adaptive trust region method has global, super-linear and quadratic convergence properties and is a numerically efficient method.

In 2011, Shi et al. [15] presented a non-monotone adaptive trust region method. This method can produce an adaptive trust region radius automatically at each iteration. This non-monotone approach and adaptive trust region radius can reduce the number of solving trust region sub-problems when reaching the same precision.

In Zhang's approach, if the trial step is not accepted, then set $p = p+1$; otherwise, let $p = 0$ and $\Delta_k = \|g_k\|\|B_k^{-1}\|$, which may be very large while r_k is very small. So, Sang and Sun [14] gave a new self-adaptive adjustment strategy for updating the trust region radius, which make full of the information at the current point. That is, given $0 \leq \eta_1 < \eta_2 < 1$, $0 < c_2 < 1 < c_1$, set

$$\Delta_{k+1} = \mu_{k+1}\|g_{k+1}\|\|B_{k+1}^{-1}\| \tag{3.5}$$

where

$$\mu_{k+1} = \begin{cases} c_1\mu_k, & r_k > \eta_2 \\ c_2\mu_k, & r_k < \eta_1 \\ \mu_k, & \eta_1 \geq r_k \leq \eta_2 \end{cases}$$

In 2015, Zhou et al. [17] used the updating rule to obtain the new radius, and they developed a non-monotone adaptive trust region method with line search based on new diagonal updating.

Non-monotone trust region methods with fixed step-size

It is well known that trust region method is a kind of important and efficient methods for nonlinear optimization. However, when a trial step is not accepted, the method need to resolve the sub-problem if we only use the trust region methods. So, many researchers presented trust region with line search methods. If the line search is expensive to compute, the methods are not good.

In the past years, Sun et al. [24] and Chen et al. [25] proposed a fixed step-length method for unconstrained optimization. In their approaches, without using line search, they computed the step-length by a formula at each iteration. Thus their methods might be practical in the cases that the line search is expensive or hard and allow a considerable saving in the number of function evaluations.

Mo et al. [23] proposed a non-monotone trust region method which combines a non-monotone technique, fixed step-length and trust region method. When the trial step is not successful, a step length is defined by a formula. They use the formula suggested by Sun and Zhang [24], Chen and Sun [25] to obtain a step-length. Most of the non-monotone method allows an increase in function value at each iteration. But in Mo's method only allows an increase in function value when trial steps are not accepted in close succession of iterations.

In 2006, Mo et al. [26] presented another non-monotone trust region method with fixed step-size. They used the non-monotone strategy proposed by Zhang et al. [8]. This non-monotone trust region with fixed step-size is efficient and robust which is showed by numerical experiments.

Non-monotone trust region methods based on conic model

The traditional non-monotone trust region methods are mostly based on quadratic model, but when the objective function has strong non-quadratic, the quadratic model methods often produce of the minimizer of the function.

In 2008, Qu et al. [27] proposed a new trust region sub-problem based on the conic model for unconstrained optimization:

$$\begin{aligned} \min \quad & c_k(d) = f_k + \frac{g_k^T d}{1-h_k^T d} + \frac{1}{2}\frac{d^T B_k d}{(1-h_k^T d)^2}, \\ s.t. \quad & 1-h_k^T d > 0, \\ & \|d\| \le \Delta_k, \end{aligned} \tag{3.6}$$

where $c_k(d)$ is called conic model which is an approximation to $f(x_k+d_k)-f(x_k)$ and h_k is the associated vector for conic model and it is usually called horizontal vector, Δ_k is conic trust region radius. The numerical results showed that the conic model method performs better than the traditional trust region methods. From then on a variety of the non-monotone trust region methods based on conic model have been presented.

The non-monotone trust region methods listed above are mostly based on quadratic model, but there are less non-monotone trust region methods based on conic model [27]; and the trust region methods based on line search techniques are all based on quadratic model. In 2009, Qu et al. [28] presented a non-monotone conic trust region method based on line search for solving unconstrained optimization. The local and global convergence properties of the non-monotone trust region method based on conic model are proved under some reasonable assumptions. Some preliminary numerical experiments and compare the performance of the new method with the performance of the method in [27]. The numerical results show that the new method performs better.

Ji et al. [29] present a new trust-region method by combining conic model method and the non-monotone technique proposed by Zhang and Hager [8]. The main difference between Qu's method [27] and the method is that in the former one the actual reduction is defined by $f_{l(k)}-f(x_k+d_k)$ which indicates that in Qu's method [27], the sequence $\{f_{l(k)}\}$ is required to be non-increasing. Whereas in the new method the actual reduction is defined by $C_k - f(x_k+d_k)$. According to (2.3), we observe that C_k is a convex combination of the function values $f(x_1), f(x_2), \dots, f(x_k)$. So, the sequence $\{C_k\}$ can be regarded as a special weighted average of the successive function values. In this method, the sequence $\{C_k\}$ is non-increasing, but the sequence of function value $\{f_k\}$ is non-monotone. With suitable assumptions, they establish the global and super-linear convergence. Numerical experiments are conducted to compare this method with Qu's method [27] indicate that the new method is superior to Qu's method [27].

In 2011, Cui et al. [30] proposed a new non-monotone conic trust region method. The algorithm is based on the Ji's method [29]. The main different is that Cui combining the line search technique. The algorithm performs a non-monotone line search to find the next iteration point when a trial step is not accepted, instead of resolving the sub-problem. Some authors presented other forms non-monotone trust region method based on conic model [31, 32, 33].

Quasi-Newton non-monotone trust region methods

Quasi-Newton techniques solve the problem (1.1) are popular, particularly whenever the matrix-valued second derivative of $f(x)$, called the Hessian and written $\nabla^2 f(x)$, is not known analytically or is prohibitively expensive to compute or store. Quasi-Newton methods use the curvature information from the current iteration, and possibly the matrix B_k to define B_{k+1}. A true quasi-Newton method will choose B_{k+1} so that

$$g_{k+1} - g_k = B_{k+1}(x_{k+1} - x_k) \tag{3.7}$$

In this way, $B_{k+1}(x_{k+1}-x_k)$ is a finite difference approximation to the derivative of $g(x)$ in the direction of $x_{k+1}-x_k$. For a practical quasi-Newton method, computing B_{k+1} should be considerably less expensive than computing $\nabla^2 f(x)$. Popular quasi-Newton methods choose $B_{k+1}=B_k+E$, where E is a matrix of low rank, usually one or two.

E. Michael Gertz [34] proposed a new trust region with line search. The algorithm not only need resolve the sub-problem, also use Wolfe line search to obtain new step at each iteration. This guarantee the sequence $\{B_k\}$ is satisfied the quasi- Newton equation and positive definiteness.

Qu et al. [38] presents a new non-monotone quasi-Newton trust-region algorithm of the conic model for the solution of unconstrained optimization problems. It is well known that in applying trust-region algorithms, the basic issue is how to solve the trust-region sub-problem efficiently. To deal with the issue, an approximate solution method is developed in this paper. Note that the approximate solution method not only is computationally cheap, but also preserves the strong convergence properties as the exact solution methods. Numerical results are shown for a number of test problems from the literature.

In 2011, Yang et al. [35] proposed a new quasi-Newton non-monotone trust region algorithm for unconstrained optimization. In this paper, the trust region radius updating used not only r_k , but also the previous ratios $\{r_{k-m},\cdots,r_k\}$, where m is some positive integer. That is,

$$r_k=\sum_{i=1}^{\min\{k,m\}}\omega_{k_i}r_{k-i+1} \tag{3.8}$$

where $\omega_{k_i}\in[0,1]$ and ω_{k_i} is satisfied $\sum_{i=1}^{\min\{k,m\}}\omega_k=1$.

In 2013, a quasi- Newton non-monotone trust region method of new conic model is proposed by Li and Qian [36]. The trust region algorithm based on new conic model is presented for unconstrained optimization by combining the non-monotonic Wolfe line search and quasi- Newton technique. The new trust region sub-problem is constructed, in which the trust region radius uses the information of g_k and B_k . The sub-problem is solved by using $c^p\left\|B_{k+1}^{-1}\right\|\left\|g_{k+1}\right\|$, $c\in(0,1)$ and p is a nonnegative integer. Therefore, instead of adjusting k , one adjusts p for each iteration. Under proper assumptions, the global convergence of the method is proved.

Other non-monotone trust region methods

All most of trust region methods have to store a symmetric matrix B_k and the algorithms are complicated relatively. So, when the scale of problem (1.1) is large, these methods may be too slow, even fail.

In 2009, based on the diagonal-sparse quasi-Newton method [12], Sun et al. [13] developed a non-monotone trust region algorithm with simple quadratic models, in which the approximation of Hessian matrix in the sub-problem is a diagonal positive definite matrix. This new trust region method is very suitable for large scale optimization problems. A non-monotone adaptive trust region line search method for large-scale unconstrained optimization was proposed by M. Ahookhosh et al. [37] in 2012.

Based on simple quadratic models of the trust region sub-problem, Zhou et al. [39] combine the trust region method with the non-monotone and adaptive techniques to propose a new non-monotone adaptive trust region algorithm for unconstrained optimization. The trust region sub-problem is very simple by using a new scale approximation of the minimizing function's Hessian. The new method needs less memory capacitance and computational complexity. The convergence results of the method are proved under certain conditions. Numerical results show that the new method is effective and attractive for large scale unconstrained problems.

Inspired by the above ideas, in 2014, Zhou et al. [16] use a new scale approximation of the minimizing function's Hessian in the trust region sub-problem, so the sub-problem form changes into the form

$$\begin{aligned}&\min q_k(d)=f_k+g_k^T d_k+\tfrac{1}{2}\gamma(x)d_k^T d_k\\&s.t.\ \ \|d\|\le\Delta_k\end{aligned} \tag{3.9}$$

and then combine the new trust region method with the non-monotone technique proposed by Zhang and Hager [8]. It only requires storage of first-order information during the process. So, this new method is especially effective for large scale problems.

CONCLUSIONS

The non-monotone trust region methods have been studied by many researchers. From above discussions, we know that the non-monotone trust region methods have many advantages and the forms of the new method are very rich. We learn that using the non-monotone strategy can ensure a better convergence and convergent rate than the methods without it. Under some mild conditions, the global convergence results of these non-monotone trust region methods have been proved. Nowadays, it is becoming one of most popular hot spot in optimization field.

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