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Killing Motion of Static Cylindrically Symmetric Spacetimes in the f(R) Gravity

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Abstract

In this study we have studied "Killing Motion of Static Cylindrically Symmetric Spacetimes in f(R) Gravity" by using algebraic and direct integration techniques. This study investigates the Killing motions of static cylindrically symmetric spacetimes with in framework of f(R) gravity, a generalization of Einstein's General Relativity. We explore the existence of Killing vector fields to understand the symmetries and conserved quantities in such spacetimes. By analysing the modified field equations, we determine the constraints imposed by f(R) gravity on the geometry and dynamics of cylindrically symmetric spacetimes. These contribute to understanding the interplay between symmetry properties and gravitational theories beyond General Relativity. The results have implications for astrophysical and cosmological models influenced by alternative gravity theories. We discussed four cases and found that the dimension of Killing vector fields is either three, four or ten.

Keywords: Killing Vector Fields, Cylindrical Symmetric, Spacetime and f(R) Gravity.

1. INTRODUCTION

The General Theory of Relativity which was proposed by Albert Einstein in 1915, has revolutionized our understanding of gravity by explaining it as the curvature of space-time caused by the mass and energy. While this theory provides a robust framework for explaining gravitational interactions on cosmic scales, it also paved the way for exploring deeper mysteries of the universe [1-4]. Among these, the concept of Dark Energy emerged, an enigmatic force which is driving the accelerated expansion of the universe, challenging our understanding of cosmology and space-time dynamics [5]. To address such complexities, several modifications have been proposed to extend General Relativity (GR). Among these, the f(R) theory, introduced by Buchdahl in 1970, stands out as a significant development. This theory modifies the Einstein-Hilbert action by introducing a functional dependence on the Ricci scalar R, offering a broader framework for gravitational dynamics [6, 7]. Despite its broader framework, obtaining direct solutions in f(R) theory remains a significant challenge. The non-linear nature of the modifications introduces complexities that make analytical solutions difficult to derive, necessitating advanced mathematical techniques and numerical approaches for solving such equations [8]. To address these challenges, various symmetry constraints, such as the Homothetic symmetry, conformal symmetry, and Killing symmetry are often employed. These symmetries help simplify the problem by reducing its degrees of freedom and making the equations more tractable. Additionally, they provide conservation laws that not only define the manifestant properties of the contents of matter but also reveal geometric structure of the space-time [9].

A notable class of space-times accordant with asymptotic uniformity includes the matter which contains constant spherically symmetric space-times. As a step beyond spherical symmetry, cylindrically symmetric spacetimes serve as a significant alternative. Levi-Civita, in 1917, derived the most general solutions for static cylindrically symmetric spacetimes, laying the foundation for further exploration in this domain [10]. Symmetry constraints, such as Killing vector fields, simplify spacetime structures by reducing degrees of freedom and providing conservation laws, aiding in the study of both matter properties and spacetime geometry [11-15].

It is important to note that a vector field K is classified as a Killing vector field if it satisfies the Killing equation, given by

$$\underset{X}{L}\, g_{ab} = g_{ab,c}X^{c} + g_{cb}X^{c} + g_{ac}X^{c} = 0 ,$$

In equation (1) $\underset{X}{L}$ represents the metric tensor's Lie derivative along the vector field X this operation measures how the metric tensor changes when one flows along the direction specified by the vector field

2. Field Equation Formulation in f(R) Gravity

Consider, in an usual coordinates, a static cylindrically symmetric spacetimes (t, r, θ, z) given by (x^0, x^1, x^2, x^3) correspondingly with line element (reference)

$$ds^2 = -e^{\nu(r)}dt^2 + dr^2 + e^{\lambda(r)}d\theta^2 + e^{\mu(r)}dz^2, \qquad (2.1)$$

In which $\nu = \nu(r), \lambda = \lambda(r), and\ \mu = \mu(r)$ are r's non-zero functions only. The above mentioned space-times (2.1) concede there linearly independents killing vector fields which are (reference)

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}. \qquad (2.2)$$

The Ricci's non-zero components of tensors for the space-times (2.1) are (reference)

$$R_{00} = \frac{1}{4}e^{\nu}\left[2\nu^{'} + \nu^{'}\lambda^{'} + \nu^{'}\mu^{'}\right],$$

$$R_{11} = -\frac{1}{4}\left[2\nu^{"} + 2\lambda^{"} + 2\mu^{"} + \nu^{'2} + \lambda^{'2} + \mu^{'2}\right],$$

$$R_{22} = -\frac{1}{4}e^{\lambda}\left[2\lambda^{"} + \lambda^{'2} + \lambda^{'}\mu^{'} + \nu^{'}\lambda^{'}\right]$$

$$R_{33} = -\frac{1}{4}e^{\mu}\left[2\mu^{"} + \mu^{'2} + \lambda^{'}\nu^{'} + \nu^{'}\mu^{'}\right], \qquad (2.3)$$

Where, the overhead prime denotes derivative with reference to r. Supposing the source of energy -momentum tensor as perfect fluid i.e. $T_{ab} = (\rho + p)s_a s_b + pg_{ab}$, where ρ is the density of matter, the pressure is denoted as p, and s_a is the four-velocity vector which is termed as $s_a = -e^{-\frac{\nu(r)}{2}\delta_a^0},$ we have $T_{00} = \rho e^{\nu}, \mathrm{T}_{11} = \rho, T_{22} = pe^{\lambda}, T_{33} = Pe^{\mu}.$
Field equation in f® gravity are (reference)

$$H_{ab} = KT_{ab,} \qquad (2.4)$$

Where $H_{ab} = F(R)R_{ab} - \frac{1}{2}f(R)g_{ab} - \Delta_a\Delta_b F(R) + g_{ab}\Box F(\mathrm{R}), \mathrm{f(R)}$ is the Ricci scalar's function R, $F(R) = \frac{d}{dr}f(R),$ the coupling constant is k, standard energy momentum tensor is T_{ab} and $\Box = \Delta^a\Delta_b$ where the covariant derivative operator is Δ_a. Using Equations. (2.1) and (2.3) in (2.9) together with non-zero components of energy -momentum tensor, we find

$$F^{"}2-\frac{F^{'}}{2}\nu^{'}+\frac{F}{4}\left[2\mu^{"}+2\lambda^{"}+\lambda^{'2}+\mu^{'2}-\nu^{'}\lambda^{'}-\nu^{'}\mu^{'}\right]+\kappa(\rho+p)=0. \quad (2.5)$$

$$\frac{F^{'}}{2}(\lambda^{'}-\nu^{'})+\frac{F}{4}\left[2\lambda^{"}-2\nu^{"}+\lambda^{'2}-\nu^{'2}+\lambda^{'}\mu^{'}-\nu^{'}\mu^{'}\right]+\kappa(\rho+p). \quad (2.6)$$

$$\frac{F^{'}}{2}(\mu^{'}-\nu^{'})+\frac{F}{4}\left[2\mu^{"}-2\nu^{"}+\mu^{"}-\nu^{'2}+\lambda^{'}\mu^{'}-\nu^{'}\lambda^{'}\right]+\kappa(\rho+p). \quad (2.7)$$

To fond the solutions of equations (2.5-2.7), by putting some restrictions on the metric coefficients together with the condition given in $F^{'}=0$, we classify space-times (2.1) (refer). Classification has the cases as follows:

(i) $\nu=\nu(r),\quad \lambda=\lambda(r),\quad \mu=cons\tan t$

(iv) $\nu=\nu(r),\ \lambda(r)=\mu(r)$

(vii) $\nu=\nu(r),\quad \mu=\lambda=cons\tan t$

(Xiii) $\rightarrow \nu=\lambda=\mu=cons\tan t$

Using $\nu=\nu(r),\ \lambda=\lambda(r),\ \mu=cons\tan t$ in equations (2.5-2.7) after some simplifications gives

$$2\lambda^{"}-2\nu^{"}-\nu^{'}\lambda^{'}-\nu^{'2}+\lambda^{'2}=0 \quad (2.8)$$

With the two unknown which are named as $\lambda\, and\, \nu$. To solve this equation, now, we suppose solution of the form $\nu=\kappa\lambda$ where $\kappa\in R-\{0,1\}$. This assumption leads to $\nu=\ln\left[\frac{4}{c_1r+c_2}\right]^4$ and $\lambda=\left[\frac{c_1r+c_2}{4}\right]^4$, therefore, space-time (2.1) takes the form, after appropriate rescaling of z

$$ds^2=-\left[\frac{4}{c_1r+c_2}\right]^4dt^2+dr^2+\left[\frac{c_1r+c_2}{c_4}\right]^4d\theta^2+dz^2. \quad (2.9)$$

Where $c_1,c_2\in R..$ It is important to mention here that we have discussed the procedure of finding the solution in only one case. Rest of the cases are similar to deal.

3 Killing Motions of Static Cylindrically Symmetric Spacetimes
Case (I)

In this case, the space-time has form:

$$ds^2=-\left[\frac{4}{c_1r+c_2}\right]^4dt^2+dr^2+\left[\frac{c_1r+c_2}{c_4}\right]^4d\theta^2+dz^2. \quad (3.1)$$

Now, we find the above space-time's Killing vector fields (3.1) by the help of equation

$$L_Xg_{ab}=0. \quad (3.2)$$

Using equation (3.1) in equation (3.2), we have ten first-order non- linear partial differential equations as follows:

$$-\frac{2c_1}{c_1r+c_2}X^1+X^0_{,0}=0 \quad (3.3$$

$$X^1_{,0}-\left[\frac{4}{c_1r+c_2}\right]X^1_{,0}=0, \quad (3.4)$$

)

$$\left[\frac{c_1 r + c_2}{4}\right]^4 X^2_{,0} - \left[\frac{4}{c_1 r + c_2}\right]^4 X^0_{,2} = 0 \qquad (3.5)$$

$$X^3_{,0} - \left[\frac{4}{c_1 r + c_2}\right]^4 X^0_{,3} = 0, \qquad (3.6)$$

$$X^1_{,1} = 0, \quad (3.7) \quad X^1_{,1} = 0, \qquad (3.7)$$

$$\left[\frac{c_1 r + c_2}{c_4}\right]^4 X^2_{,1} + X^1_{,2} = 0, \qquad (3.8)$$

$$X^1_{,3} + X^3_{,1} = 0, \qquad (3.9)$$

$$\frac{2c_1}{c_1 r + c_2} X^1 + X^2_{,2} = 0, \qquad (3.10)$$

$$X^3_{,2} + \left[\frac{c_1 r + c_2}{4}\right]^4 X^2_{,3} = 0, \qquad (3.11)$$

$$X^3_{,3} = 0. \qquad (3.12)$$

From equations (3.7) and (3.12), we have $X^1 = A^1(t,\theta,z)$ *and* $X^3 = A^2(t,r,\theta)$ in which $A^1(t,\theta,z)$ *and* $A^2(t,r,\theta)$ are integration's functions need to determine. Now, using the value of X^1 in equation (3.3), we get

$$X^0 = \frac{2c_1}{c_1 r + c_2}\int A^1(t,\theta,z)dt + A^3(r,\theta,z)$$

Where $A^3(r,\theta,z)$ is another function of integration. On the same pattern, if we substitute the value of X^1 in equation (3.10) we obtained the value of X^2 which is

$$X^2 = -\frac{2c_1}{c_1 r + c_2}\int A^1(t,\theta,z)d\theta + A^4(t,r,z),$$

Where $A^4(t,r,z)$ is another function of integration. Now, we have the following initial system:

$$\left.\begin{aligned} X^0 &= \frac{2c_1}{c_1 r + c_2}\int A^1(t,\theta,z) + A^3(r,\theta,z) \\ X^1 &= A^1(t,\theta,z) \\ X^2 &= -\frac{2c_1}{c_1 r + c_2}\int A^1(t,\theta,z)d\theta + A^4(t,r,z) \\ X^3 &= A^2(t,r,\theta) \end{aligned}\right\}, \qquad (3.13)$$

After some tedious calculations and avoiding lengthy calculations we get the following result

$$X^0 = c_{11}, X^1 = 0, X^2 = c_9, X^3 = c_6, \qquad (3.14)$$

In this case, the generators of the Killing algebra are represented by X.

$$X_1 = \frac{\partial}{\partial z}, X_2 = \frac{\partial}{\partial \theta} \text{ and } X_3 = \frac{\partial}{\partial t}$$

Where $c_6, c_9, c_{11} \in \Re$

Case (II)
In this case, the space-time metric has the form:

$$ds^2 = -(c_1 r + c_2)^6 dt^2 + dr^2 + (c_1 r + c_2)^3 d\theta^2 + (c_1 r + c_2) dz^2. \quad (3.15)$$

Now, we find above space-time's Killing vectors fields (3.2.1) with the help of equation (3.2). Using equation (3.15) in equation (3.2), we have the ten first-order non-linear partial differential equations as follows:

$$\frac{3c_1}{c_1 r + c_2} X^1 + X^0_{,0} = 0 \quad (3.16)$$

$$X^1_{,0} - (c_1 r + c_2)^6 X^0_{,1} = 0, \quad (3.17)$$

$$X^2_{,0} - (c_1 r + c_2)^3 X^0_{,2} = 0, \quad (3.18)$$

$$X^3_{,0} - (c_1 r + c_2)^3 X^0_{,3} = 0, \quad (3.19)$$

$$X^1_{,1} = 0, \quad (3.20)$$

$$X^1_{,2} + (c_1 r + c_2)^3 X^2_{,1} = 0, \quad (3.21)$$

$$X^1_{,3} + (c_1 r + c_2)^3 X^3_{,1} = 0, \quad (3.22)$$

$$\frac{3c_1}{2(c_1 r + c_2)} X^1 + X^2_{,2} = 0, \quad (3.23)$$

$$X^3_{,2} + X^2_{,3} = 0, \quad (3.24)$$

$$\frac{3c_1}{2(c_1 r + c_2)} X^1 + X^3_{,3} = 0 \quad (3.25)$$

From equation (3.20), we have $X^1 = A^1(t, \theta, \mathrm{z}),$ where $A^1(t, \theta, \mathrm{z})$ is function of integration need to be determined. Similarly after solving these equation we get the initial system

$$\left.\begin{aligned} X^0 &= -\frac{3c_1}{c_1 r + c_2} \int A^1(t, \vartheta, z) dt + A^2(r, \theta, z) \\ X^1 &= A^1(t, \theta, z) \\ X^2 &= -\frac{3c_1}{2(c_1 r + c_2)} \int A^1(t, \vartheta, z) d\theta + A^3(r, \theta, z) \\ X^3 &= -\frac{3c_1}{2(c_1 r + c_2)} \int A^1(t, \vartheta, z) dz + A^4(\mathrm{t}, \mathrm{r}, \theta) \end{aligned}\right\} \quad (3.26)$$

Where $c_1, c_2 \in \Re$ and $A^2(r, \theta, z)$, $A^3(r, \theta, z)$, $A^4(\mathrm{t}, \mathrm{r}, \theta)$ are functions of integration. Now, by omitting the lengthy calculations, we present the final results:

$$\left.\begin{aligned} X^0 &= c_{18} \\ X^1 &= 0 \\ X^2 &= c_6 \\ X^3 &= c_{16} \end{aligned}\right\}, \quad (3.27)$$

In this case, the X denotes the generators of the Killing algebra.

$$X_1 = \frac{\partial}{\partial \theta}, X_2 = \frac{\partial}{\partial z} \, and \, X_3 = \frac{\partial}{\partial t}$$

$$where \, c_6, c_{16}, c_{18} \in \Re$$

Case (III)

In this case, the space-time has following form:

$$ds^2 = -\beta dt^2 + dr^2 + \beta d\theta^2 + \delta dz^2 \qquad (3.28)$$

Now, we find above space-time's KVFs (3.38) with the help of equation (3.2). Using equation (3.28) in equation (3.2), we have the following ten firs-order non-linear differential equations:

$$X^0_{,0} = 0 \quad (3.29) \qquad X^1_{,0} - \beta X^0_{,1} = 0, \quad (3.30) \qquad \beta X^2_{,0} - \beta X^0_{,2} = 0, \quad (3.31)$$

$$\delta X^3_{,0} - \beta X^0_{,3} = 0, \quad (3.32) \qquad X^1_{,1} = 0, \quad (3.33) \qquad X^1_{,2} + \beta X^2_{,1} = 0, \quad (3.34)$$

$$X^1_{,3} + \delta X^3_{,1} = 0, \quad (3.35) \qquad X^2_{,2} = 0 \quad (3.35) \qquad \beta X^2_{,3} + \delta X^3_{,2} = 0, \quad (3.36)$$

$$X^3_{,3} = 0 \quad (3.37)$$

From equation (3.29), (3.33), (3.35) and (3.3.7) we get the following initial system:

$$\left.\begin{aligned} X^0 &= A^1(r,\theta,z) \\ X^1 &= A^2(t,\theta,z) \\ X^2 &= A^3(t,r,z) \\ X^3 &= A^4(t,r,\theta) \end{aligned}\right\}, \qquad (3.38)$$

By skipping the lengthy calculations, we arrive at directly at the final system:

$$\left.\begin{aligned} X^0 &= \frac{r}{\beta} c_4 + \theta c_6 + z c_7 + c_8 \\ X^1 &= t c_4 - \theta \beta c_{15} + c_{17} \\ X^2 &= t c_6 + r c_{15} + z c_{22} + c_{23} \\ X^3 &= \frac{t\beta}{\delta} c_7 + \theta c_{24} + c_{25} \end{aligned}\right\}, \qquad (3.39)$$

In this scenario, the Killing algebra generators are labelled as X.

$$X_1 = \frac{r}{\beta}\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}, X_2 = \theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}, X_3 = z\frac{\partial}{\partial t} + \frac{t\beta}{\delta}\frac{\partial}{\partial z}$$

$$X_4 = \frac{\partial}{\partial t}, X_5 = -\theta\beta\frac{\partial}{\partial r} + r\frac{\partial}{\partial \theta}, X_6 = \frac{\partial}{\partial r}, X_7 = z\frac{\partial}{\partial \theta}$$

$$X_8 = \frac{\partial}{\partial \theta}, X_9 = \theta\frac{\partial}{\partial z}, X_{10} = \frac{\partial}{\partial z}$$

$$where \, c_4, c_6, c_7, c_8 \in \Re$$

Case (IV)

The spacetime in this case has the form:

$$ds^2 = -r^2 dt^2 + dr^2 + d\theta^2 + dz^2 \qquad (3.40)$$

Now, we find above space-time's Killing vectors fields (3.40) with the help of equation (3.2), we have the ten first-order non-linear partial differentia equations as follows:

$$\frac{1}{r}X^1 + X^0_{,0} = 0 \quad (3.41) \qquad X^1_{,0} - r^2 X^0_{,2} = 0, \quad (3.42) \qquad X^2_{,0} - r^2 X^0_{,2} = 0, \quad (3.43)$$

$$X^3_{,0} - r^2 X^0_{,3} = 0, \quad (3.44) \qquad X^1_{,1} = 0, \quad (3.45) \qquad X^1_{,2} + X^2_{,1} = 0, \quad (3.46)$$

$$X^2_{,2} = 0, \quad (3.47) \qquad X^1_{,3} + X^3_{,1} = 0, \quad (3.48) \qquad X^2_{,3} + X^2_{,3} = 0, \quad (3.49)$$

$$X^3_{,3} = 0. \quad (3.50)$$

From equation (3.45), (3.47),(3.50) and (3.41) after solving we get the initial system:

$$\left.\begin{aligned} X^0 &= -\frac{1}{r}\int A^1(t,\theta,z)dt + A^4(r,\theta,z) \\ X^1 &= A^1(t,\theta,z) \\ X^2 &= A^2(t,r,z) \\ X^3 &= A^3(t,r,\theta) \end{aligned}\right\} \quad (3.51)$$

By avoiding extensive calculations we directly reach the final system:

$$\left.\begin{aligned} X^0 &= c_3 t + c_7\theta + c_8 z + c_9 \\ X^1 &= c_5 r + c_6 t + \theta c_{10} + c_{11} z + c_{12} \\ X^2 &= c_5\theta + c_7 t + z c_{13} + c_{14} \\ X^3 &= c_5 z - \frac{1}{\beta}c_{11} r + c_{15} \end{aligned}\right\}, \quad (3.52)$$

Generator form

$$X_1 = t\frac{\partial}{\partial t}, X_2 = r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z}, X_3 = t\frac{\partial}{\partial r}, \quad X_4 = \theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}, \quad X_5 = z\frac{\partial}{\partial t}, \quad X_6 = \frac{\partial}{\partial t}$$

$$X_7 = \theta\frac{\partial}{\partial r}, X_8 = z\frac{\partial}{\partial r} - \frac{r}{\beta}\frac{\partial}{\partial z}, X_9 = \frac{\partial}{\partial r}, X_{10} = z\frac{\partial}{\partial \theta}, \quad X_{11} = \frac{\partial}{\partial \theta}, X_{12} = \frac{\partial}{\partial z}$$

CONCLUSIONS

In this study, we have discussed the static cylindrically symmetric space-time's Killing vector fields in f (R) gravity. Four cases exist in this study. Following results have been obtained by studying every case:

In case (i) 3 is the Killing vector field's dimension.

In case (ii) the dimension of Killing vector field is 4.

In the cases (iii) and (iv), dimension of Killing vector fields is 10.

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